**Numerical Analysis I**  
*(4th Homework Assignment)*

**Exercise 13** *(Gradient method and semi-iterative Richardson iteration)*

Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite with the extreme eigenvalues $\lambda := \lambda_{\text{min}}(A)$, $\Lambda := \lambda_{\text{max}}(A)$ and let $\kappa := \Lambda / \lambda$ be the spectral condition of $A$. Further, let $b \in \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$.

(i) Consider the semi-iterative Richardson iteration

$$y^{m+1} = y^m + \Theta_{m+1}(b - Ay^m) , \ m \geq 0 .$$

Specify the values of $y^0 \in \mathbb{R}^n$ and $\Theta_{m+1}$ for which the sequence of iterates $(y^m)_N$ corresponds to the sequence $(x^m)_N$ obtained by the gradient method.

(ii) Show that for any initial vector $x^0 \in \mathbb{R}^n$ the sequence $(x^m)_N$, obtained by the gradient method, converges to $x^* := A^{-1}b$. Moreover, verify the estimates

$$F(x^m) - F(x^*) \leq \left( \frac{\kappa - 1}{\kappa + 1} \right)^{2m} [F(x^0) - F(x^*)] ,$$

$$\|x^m - x^*\|_A \leq \left( \frac{\kappa - 1}{\kappa + 1} \right)^{m} \|x^0 - x^*\|_A ,$$

where $F(x) := \frac{1}{2} < Ax, x > - < b, x >$.

[Hint: Utilize Theorem 1.9 as presented in class and take advantage of the fact that $x^m$ minimizes the error $\|x^m - x^*\|_A$.]

**Exercise 14** *(Convergence of the cg method)*

Assume that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite with $p < n$ different eigenvalues. Consider the linear system $Ax = b \in \mathbb{R}^n$ and show that in case of exact computations, for any initial iterate $x^0 \in \mathbb{R}^n$ the cg method terminates after at most $p$ steps with the solution $A^{-1}b$.

[Hint: Apply the the Cayley-Hamilton theorem together with estimates for the iteration error.]
Exercise 15 (Preconditioned cg method)

The numerical solution of two-point boundary value problems often gives rise to a symmetric matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$, where

$$a_{ij} = \begin{cases} 2 & \text{f"ur } i = j \\ -1 & \text{f"ur } |i - j| = 1 \\ 0 & \text{f"ur } |i - j| \geq 2 \end{cases} , \quad 1 \leq i, j \leq n .$$

(i) Show that $A$ is positive definite.

(ii) Let $D := \text{diag}(A)$ be the diagonal of $A$ and denote by $L$ the lower triangular part so that $A = L + D + L^T$.

Consider the linear system $Ax = b$, $b \in \mathbb{R}^n$ and show that the preconditioned cg method with preconditioner $C := EE^T$, where $E := \frac{1}{2}D + L$, converges after at most two steps.

[Hint: Observe that the convergence properties of the preconditioned cg method correspond to the original cg method applied to the transformed matrix $\tilde{A} := E^{-1}AE^{-T}$ and apply the result of Exercise 14.]

Exercise 16 (cg method for the discrete Laplacian)

The discretization of the 2D Laplacian by finite differences with respect to a uniform grid results in a linear algebraic system with the symmetric coefficient matrix $A = (a_{\ell j})_{\ell,j=0}^{n-1} \in \mathbb{R}^{n \times n}$, $n = k^2$, $k \in \mathbb{N}$, where

$$a_{\ell j} = \begin{cases} 4 & , \ell = j \\ -1 , & (\ell \text{DIV} k) = (j \text{DIV} k) \wedge |\ell - j| = 1 \\ -1 , & |\ell - j| = k \\ 0 , & \text{otherwise} \end{cases} , \quad 1 \leq \ell, j \leq n - 1 .$$

Here, $\ell \text{DIV} k$ stands for the integer part of $\ell/k$, whereas $\ell \text{MOD} k \in \{0, \ldots, k-1\}$ denotes the rest of the division.

(i) Represent $A$ as a $k \times k$ block matrix and specify the individual blocks.

(ii) A vector $x = (x_0, \ldots, x_{n-1})^T \in \mathbb{R}^n$ can be stored in a $k \times k$ array $\hat{x}$ as follows:

$$x_\ell = \hat{x}[\ell \text{DIV} k, \ell \text{MOD} k] , \quad 0 \leq \ell \leq n - 1 .$$

Using this ‘two-dimensional’ ordering of the components of the vector, the action of the matrix $A$ can be easily explained. Describe by means of a two-dimensional drawing how the components of $\hat{x}$ change under the mapping represented by the matrix $A$.

[Hint: $\hat{x}$ is a $k \times k$ array of certain values. In case of multiplication by $A$, a new value is assigned to each grid point which can be written as a weighted sum of other values. The weights in this sum correspond to the entries of $A$.]
(iii) Assume that for $1 \leq \mu, \nu \leq k$ the vectors $e_{\mu,\nu} \in \mathbb{R}^n$ are stored in the $k \times k$ arrays $\hat{e}_{\mu,\nu}$ of complex numbers. In this form, they can be represented according to

$$\hat{e}_{\mu,\nu}[i,j] := \sin\left(\pi\mu \frac{i+1}{k+1}\right) \sin\left(\pi\nu \frac{j+1}{k+1}\right), \quad 0 \leq i, j \leq k - 1.$$ 

Show that $e_{\mu,\nu}$ is an eigenvector of $A$ corresponding to the eigenvalue

$$\lambda_{\mu,\nu} = 4\left(\sin^2\left(\frac{\pi}{2} \frac{\mu}{k+1}\right) + \sin^2\left(\frac{\pi}{2} \frac{\nu}{k+1}\right)\right).$$

(iv) Compute $\lambda_{\text{max}}(A) := \max\{|\lambda|; \lambda \in \sigma(A)\}$, $\lambda_{\text{min}}(A) := \min\{|\lambda|; \lambda \in \sigma(A)\}$ and the spectral condition number

$$\kappa(A) := \frac{\lambda_{\text{max}}(A)}{\lambda_{\text{min}}(A)}.$$

(v) Show that $A$ is positive definite.

(vi) Derive an estimate for the convergence rate of the cg method in the $A$ energy norm. How does the convergence behave in terms of $k$?

Delivery of the homework at latest on October 2, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.