Exercise 17 (Reformulation of linear least squares problems)
Let \( A \in \mathbb{R}^{m \times n} \), \( m > n \), \( \text{rank} \, A = n \), \( b \in \mathbb{R}^{m} \). The linear least squares problem
\[
(\ast) \quad \|Ax - b\|_2 = \min
\]
can be formulated as the linear algebraic system
\[
(\ast\ast) \quad \begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix},
\]
where \( I_m \) stands for the \( m \times m \) unit matrix and \( r := b - Ax \in \mathbb{R}^{m} \) is the residual.

(i) Using the normal equations, show that the component \( x \) of the solution of (\ast\ast) solves the linear least squares problem (\ast).

(ii) Given a decomposition of \( A \) according to
\[
(\dagger) \quad A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}
\]
with an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \) and a regular upper triangular matrix \( R \in \mathbb{R}^{n \times n} \) show that by orthogonal row and column transformations the linear system
\[
\begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]
can be transformed to the form
\[
\begin{pmatrix} I_n & 0 & R \\ 0 & I_{m-n} & 0 \\ R^T & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ d \\ z \end{pmatrix} = \begin{pmatrix} f_1 \\ d \\ g \end{pmatrix}.
\]
Exercise 18 (Least squares polynomial approximation)
Let \( f : [0,1] \rightarrow \mathbb{R} \) be a continuous function. We are looking for a polynomial
\[
P^{(m)}(t) := \sum_{k=0}^{n-1} x^{(m)}_k t^k,
\]
such that for \( t_\ell := \frac{\ell}{m}, m > n \), there holds:
\[
\frac{1}{m} \sum_{\ell=0}^{m} (f(t_\ell) - P^{(m)}(t_\ell))^2 = \min.
\]

(i) Formulate the problem as a linear least squares problem
\[
\| A^{(m)} x^{(m)} - b^{(m)} \|_2 = \min
\]
with respect to the coefficients \( x^{(m)} = (x^{(m)}_0, \ldots, x^{(m)}_{n-1}) \). In particular, specify \( A^{(m)} \) and \( b^{(m)} \).

(ii) Determine a linear system such that its solution is the limit of the sequence \( x^{(m)} \) für \( m \rightarrow \infty \).

Exercise 19 (Updating least squares problems)
Let \( A \in \mathbb{R}^{m \times n}, m > n \), \( \text{rank} A = n \) and \( u \in \mathbb{R}^m, v \in \mathbb{R}^n \). Consider
\[
\hat{A} := A + uv^T
\]
and show: If \( \text{rank} \hat{A} = n \) and if
\[
A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}
\]
is the QR-decomposition of \( A \), then the QR-decomposition of \( \hat{A} \) is given by
\[
\hat{A} = \hat{Q} \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix},
\]
where
\[
\hat{Q} = QU, \quad U = F_{m-1} \cdots F_1 \hat{F}_1 \cdots \hat{F}_n
\]
with suitable Givens rotations \( F_k, 1 \leq k \leq m-1 \), \( \hat{F}_\ell, 1 \leq \ell \leq n \).

[Hint: First, compute \( F_k, 1 \leq k \leq m-1 \) such that
\[
F_1^T \cdots F_{m-1}^T Q^T u = \alpha e_1, \quad \alpha = \pm \| Q^T u \|_2.
\]
Show that
\[
\hat{H} = \hat{F}_1^T \cdots \hat{F}_n^T \begin{pmatrix} R \\ 0 \end{pmatrix} + (Q^T u) v^T
\]
is an upper Hessenberg matrix. Then, choose \( \hat{F}_\ell, 1 \leq \ell \leq n, \) correspondingly.]
**Exercise 20 (Downdating least squares problems)**

For linear least squares problems, 'downdating' refers to the effect of eliminating an observation:

Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, with rank $A = n$ and let

$$ A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} $$

be the QR-decomposition of $A$. Partition $A$ according to

$$ A = \begin{pmatrix} a_1^T \\ \tilde{A} \end{pmatrix}, $$

where $\tilde{A} \in \mathbb{R}^{(m-1) \times n}$ and $a_1^T$ is the first row of $A$ and show:

The QR-decomposition of $\tilde{A}$ is given by

$$ \tilde{A} = \tilde{Q} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}, \quad \tilde{Q} = Q F_{m-1} \ldots F_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{Q} \end{pmatrix}, \quad |\alpha| = 1 $$

with suitably chosen Givens rotations $F_k$, $1 \leq k \leq m - 1$.

[Hint: Either apply Exercise 19 or show that the 'downdating' problem is equivalent to the QR-decomposition of the extended matrix $(e_1, A) \in \mathbb{R}^{m \times (n+1)}$.]

Delivery of the homework at latest on October 16, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.