Numerical Mathematics I (6th Homework Assignment)

Exercise 21 (Regula falsi of higher order)
Assume that \( f \in C^1([a,c]), 0 < a < c \), has a simple zero \( x^* \in [a,c] \) and that \( f(a)f(c) < 0 \). Let \( b \) such that \( a < b < c \) and consider the quadratic polynomial \( p \in P_2(\mathbb{R}) \) interpolating \( f \) in \( a, b, c \), i.e. \( p(z) = f(z), z \in \{a, b, c\} \).

(i) Show that \( p \) has exactly one zero in \([a,c]\).

(ii) Denote by \( \text{REAL zero}(a, b, c, f) \) the function which computes the zero of \( p \) for given \( a, b, c \). Using this function, construct an iterative algorithm which provides an approximation of \( x^* \) up to machine accuracy \( \varepsilon \).

Exercise 22 (Newton’s method for the computation of the \( m \)-th root)

(i) For \( m \in \mathbb{N} \), the positive \( m \)-th root \( x^* > 0 \) of \( a > 0 \) is the solution of the equation \( f(x) = 0 \) where \( f(x) := x^m - a \).

Assume \( m \geq 2 \) and determine the Newton iteration

\[
x^{(k+1)} = \Phi(x^{(k)})
\]

for the computation of \( x^* \).

(ii) Let \( f \) be a scalar real-valued function with zero \( x^* \) and denote by \( x^{(k)}, k \in \mathbb{N} \), the iterates obtained by Newton’s method. Show that there holds

\[
x^* - x^{(k+1)} = -\frac{1}{2} \frac{f''(x^{(k)}) + \theta(x^+ - x^{(k)})}{f'(x^{(k)})} (x^* - x^{(k)})^2,
\]

where \( 0 \leq \theta \leq 1 \).

(iii) For the particular function \( f \) from part (i) compute the asymptotic error constant

\[
C := \frac{1}{2} \left| \frac{f''(x^*)}{f'(x^*)} \right|.
\]
What is its meaning with regard to the convergence of Newton’s method close to the zero $x^*$?

**Exercise 23 (Perturbation lemma)**

Given $B \in \mathbb{R}^{n \times n}$ and a submultiplicative matrix norm $\| \cdot \|$, assume $\|B\| < 1$. Show that the matrix $I - B$ is regular with

$\|(I - B)^{-1}\| \leq \frac{1}{1 - \|B\|}$.

**Exercise 24 (Affine invariant convergence result for Newton’s method)**

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, $D$ convex, be continuously differentiable. Assume that $x^* \in D$ is a zero of $F$ with regular Jacobian matrix $F'(x^*)$. Moreover, assume that for some $\omega^* > 0$ the following affine invariant Lipschitz condition holds true

$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq \omega^* \|y - x\|$, $x, y \in D$,

and that

$B_\rho(x^*) := \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq \rho\} \subset D$ for $\rho := \|x^{(0)} - x^*\| < \frac{2}{3\omega^*}$

for some initial value $x^{(0)}$.

If $\{x^{(k)}\}_{k \in \mathbb{N}_0}$ is the sequence of Newton iterates, show that $x^{(k)} \in B_\rho(x^*)$, $k \in \mathbb{N}_0$, and that $x^{(k)} \to x^*$ as $k \to \infty$.

Moreover, show that $x^*$ is the unique zero of $F$ in $D$.

[Hint: Use induction on $k$ and apply the perturbation lemma from Exercise 23 to verify that $F'(x^{(k)}), k \in \mathbb{N}_0$, is regular and $\|F'(x^{(k)})^{-1}F'(x^*)\| \leq 3$.]

Delivery of the homework at latest on October 30, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.