Chapter 3 Conforming Finite Element Methods

3.1 Foundations

3.1.1 Ritz-Galerkin Method

Let $V$ be a Hilbert space, $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ a bounded, $V$-elliptic bilinear form and $\ell : V \to \mathbb{R}$ a bounded linear functional. We want to approximate the variational equation:

$$a(u, v) = \ell(v), \quad v \in V.$$  

We recall that the Lax-Milgram Lemma ensures the existence and uniqueness of a solution of (3.1).

Now, given a finite-dimensional subspace $V_h \subset V$, $\dim V_h = n_h$, the Ritz-Galerkin method is to approximate (3.1) by its restriction to $V_h$:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h), \quad v_h \in V_h.$$  

Again, by the Lax-Milgram Lemma we know that (3.2) has a unique solution $u_h \in V_h$.

It is easy to see that the Ritz-Galerkin method gives rise to a linear algebraic system, once we specify a basis of $V_h$. Therefore, let us assume that $(\varphi_h^{(i)})_{i=1}^{n_h}$ is a basis of $V_h$, i.e.,

$$V_h = \text{span}(\varphi_h^{(1)}, \ldots, \varphi_h^{(n_h)}).$$  

Then, the solution $u_h \in V_h$ of (3.2) can be represented as a linear combination of the basis functions according to

$$u_h = \sum_{j=1}^{n_h} \alpha_j \varphi_h^{(j)}.$$  

Apparently, $u_h$ as given by (3.4) satisfies (3.2) if and only if

$$\sum_{j=1}^{n_h} a(\varphi_h^{(i)}, \varphi_h^{(j)}) \alpha_j = \ell(\varphi_h^{(i)}), \quad 1 \leq i \leq n_h.$$  

Obviously, (3.5) represents a linear algebraic system

$$A_h \alpha_h = b_h.$$  

(3.6)
in the unknown vector $\alpha_h = (\alpha_1, ..., \alpha_{n_h})^T$, where the stiffness matrix $A_h \in \mathbb{R}^{n_h \times n_h}$ and the load vector $b_h \in \mathbb{R}^{n_h}$ are given by

$$A_h := \begin{pmatrix}
a(\varphi_h^{(1)}, \varphi_h^{(1)}) & \cdots & a(\varphi_h^{(n_h)}, \varphi_h^{(n_h)}) \\
\vdots & \ddots & \vdots \\
a(\varphi_h^{(1)}, \varphi_h^{(n_h)}) & \cdots & a(\varphi_h^{(n_h)}, \varphi_h^{(n_h)})
\end{pmatrix}$$

and

$$b^{(i)}_h := \ell(\varphi_h^{(i)}) \quad 1 \leq i \leq n_h.$$ 

There are basically two major aspects with regard to the construction of appropriate finite-dimensional subspaces $V_h$:

- The efficient numerical solution of the linear algebraic system (3.6).
- The accuracy of the approximation of the solution $u \in V$ of (3.1) by the solution $u_h \in V_h$ of (3.2).

### 3.1.2 Céa’s Lemma

Céa’s Lemma tells us that under the assumptions of the Lax-Milgram Lemma the accuracy of the solution $u \in V$ of (3.1) by the solution $u_h \in V_h$ of (3.2) is as good as the best approximation of $u \in V$ by a function in $V_h$ which reduces this issue to a problem of approximation theory.

It is based on the observation that the error $u - u_h$ is a-orthogonal to $V_h$, i.e.,

$$a(u - u_h, v_h) = 0 \quad v_h \in V_h ,$$

a property which is referred to as Galerkin orthogonality.

**Definition 3.1** Elliptic projection

If the bilinear form $a(\cdot, \cdot)$ is symmetric and V-elliptic, it defines an inner product on $V$, the energy inner product. Then, (3.9) states that the solution $u_h \in V_h$ of (3.2) is the projection of the solution $u \in V$ of (3.1) onto $V_h$ which is called the **elliptic projection**.

**Lemma 3.1** Céa’s Lemma

Assume that $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is a bounded, V-elliptic bilinear form, i.e., there exist positive constants $C$ and $\alpha$ such that

$$|a(u, v)| \leq C \|u\|_V \|v\|_V , \quad u, v \in V ,$$

$$a(u, u) \geq \alpha \|u\|_V^2 , \quad u \in V .$$

Assume further that $\ell \in V^*$ is a bounded linear functional and that $u \in V$ and $u_h \in V_h$ are the unique solutions of (3.1) and (3.2), respectively.
Then, there holds

\[(3.12) \quad \|u - u_h\|_V \leq C \alpha \inf_{v_h \in V_h} \|u - v_h\|_V .\]

**Proof.** Using the V-ellipticity and boundedness of \(a(\cdot, \cdot)\) as well as the Galerkin orthogonality (3.9), we find that for any \(v_h \in V_h\)

\[
\alpha \|u - u_h\|^2_V \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \leq 0
\]

\[
\leq C \|u - u_h\|_V \|u - v_h\|_V ,
\]

from which we readily deduce (3.12).

### 3.2. Triangulations

The construction of conforming finite element spaces is based on a suitable partition of the computational domain.

**Definition 3.2 Triangulation**

Let \(\Omega \subset \mathbb{R}^d\) be a bounded domain with a Lipschitz continuous boundary \(\Gamma = \partial \Omega\). A triangulation \(T_h\) of \(\overline{\Omega}\) is a partition of \(\overline{\Omega}\) into a finite number of subsets \(K\), called **finite elements**, such that

\[(T_h1) \quad \overline{\Omega} = \bigcup_{K \in T_h} K,
(T_h2) \quad K = \overline{K} , \ K^c \neq \emptyset , \ K \in T_h ,
(T_h3) \quad K_1^c \cap K_2^c = \emptyset \ \text{for all} \ K_1, K_2 \in T_h , \ K_1 \neq K_2 ,
(T_h4) \quad \partial K , \ K \in T_h \ \text{is Lipschitz continuous}.
\]

We consider two types of elements: **d-simplices** and **d-rectangles**.

**Definition 3.3 d-simplex**

A **d-simplex** \(K\) in \(\mathbb{R}^d\) is the convex hull of \(d + 1\) points \(a_j = (a_{ij})_{i=1}^{d} \in \mathbb{R}^d\):

\[
K = \{ x = \sum_{j=1}^{d+1} \lambda_j a_j \mid 0 \leq \lambda_j \leq 1 , \ \sum_{j=1}^{d+1} \lambda_j = 1 \}.
\]

The simplex \(K\) is called **non degenerate**, if any point \(x \in \mathbb{R}^d\) can be uniquely represented in the form

\[
x = \sum_{j=1}^{d+1} \lambda_j a_j , \quad \lambda_j \in \mathbb{R} , \ \sum_{j=1}^{d+1} \lambda_j = 1 .
\]
Remark 3.1 Non degeneracy of a d-simplex

The non degeneracy of a d-simplex is related to the unique solvability of the linear system

\[
\begin{pmatrix}
\ldots & a_{1,d+1} \\
\ldots & \ldots & \ldots \\
1 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_d \\
\lambda_{d+1}
\end{pmatrix}
=:
\begin{pmatrix}
x_1 \\
\vdots \\
x_d \\
1
\end{pmatrix}
\]

(3.13)

Obviously, \( K \) is non degenerate if and only if the matrix \( A \) is regular.

Figure 3.1: Triangle (d=2) and tetrahedron (d=3)

Definition 3.3 Vertices, edges, and faces

The points \( a_j, 1 \leq j \leq d+1 \), of a d-simplex \( K \) are called vertices. An m-dimensional face of a d-simplex \( K, 0 \leq m \leq d \), is an m-simplex whose vertices correspond to vertices of \( K \). A 1-dimensional face is referred to as an edge. For \( D \subseteq \Omega \), we refer to \( V_h(D), F_h(D), \) and \( E_h(D) \) as the sets of vertices, \((d-1)\)-dimensional faces and edges of \( T_h \), respectively. The d-simplex \( \hat{K} \) with vertices \( \hat{a}_1 = (0, \ldots, 0)^d \) and \( \hat{a}_{i+1} = e_i, 1 \leq i \leq d \), is referred to as the unit d-simplex.
Definition 3.4 Barycentric coordinates, center of gravity
The barycentric coordinates $\lambda_j, 1 \leq j \leq d + 1$, of a point $x \in \mathbb{R}^d$ with respect to the $d + 1$ vertices $a_j$ of a non degenerate d-simplex $K$ are the components of the unique solution of the linear system (3.13). The center of gravity $x_S$ of a non degenerate d-simplex $K$ is the point with

$$\lambda_j(x_S) = \frac{1}{d + 1}, \quad 1 \leq j \leq d + 1.$$ 

Lemma 3.2 Affine transformation
Any non degenerate d-simplex $K \subset \mathbb{R}^d$ is the image of the unit d-simplex $\hat{K}$ under an affine transformation

$$F_K : \mathbb{R}^d \rightarrow \mathbb{R}^d,$$ $$\hat{x} \mapsto F_K(\hat{x}) = B_K \hat{x} + b_K$$

with a nonsingular matrix $B_K \in \mathbb{R}^{d \times d}$ and $b_K \in \mathbb{R}^d$.

Definition 3.5 Simplicial triangulation
A triangulation $\mathcal{T}_h$ of a polyhedral domain $\Omega \subset \mathbb{R}^d$ is called a simplicial triangulation, if its elements $K$ are d-simplices.

Definition 3.6 d-rectangle
A d-rectangle $K$ in $\mathbb{R}^d$ is the tensor product of $d$ intervals $[c_i, d_i], c_i \leq d_i, 1 \leq i \leq d$, i.e.,

$$K = \prod_{i=1}^{d} [c_i, d_i] = \{ x = (x_1, ..., x_d)^T \mid c_i \leq x_i \leq d_i, \quad 1 \leq i \leq d \}.$$ 

A d-rectangle $K$ is said to be non degenerate, if $c_i < d_i, 1 \leq i \leq d$. The d-rectangle $\hat{K} := [0, 1]^d$ is called the unit d-rectangle (unit cube) in $\mathbb{R}^d$.

Figure 3.2: 2-Rectangle (d=2) and 3-rectangle (d=3)
**Definition 3.7 Vertices, edges, and faces**

The points \( a_j, 1 \leq j \leq 2^d \), of a d-rectangle \( K \) given by
\[
a_j = (a_{j1}, \ldots, a_{jd})^T, \quad a_{ji} = c_i \text{ or } a_{ji} = d_i, \quad 1 \leq i \leq d,
\]
are called vertices.

An \( m \)-dimensional face of a d-rectangle \( K \), \( 1 \leq m \leq d - 1 \), is an \( m \)-rectangle whose vertices correspond to vertices of \( K \).

A 1-dimensional face is referred to as an edge.

For \( D \subseteq \overline{\Omega} \), we refer to \( V_h(D) \), \( F_h(D) \), and \( E_h(D) \) as the sets of vertices, \((d - 1)\)-dimensional faces and edges of \( T_h \), respectively.

**Lemma 3.3 Diagonal affine transformation**

Any non degenerate d-rectangle \( K \subset \mathbb{R}^d \) is the image of the unit d-rectangle (unit cube) \( \hat{K} \) under a diagonal affine transformation
\[
F_K : \mathbb{R}^d \to \mathbb{R}^d
\]
\[
\hat{x} \mapsto F_K(\hat{x}) = B_K \hat{x} + b_K
\]
with a nonsingular diagonal matrix \( B_K = (b_{ii})_{i=1}^d \) and \( b_K \in \mathbb{R}^d \).

**Definition 3.8 Rectangular triangulation**

A triangulation \( T_h \) of a rectangular domain \( \Omega \subset \mathbb{R}^d \) is called a rectangular triangulation, if its elements \( K \) are d-rectangles.

### 3.3 Local specification of finite elements

With respect to a triangulation \( T_h \) of the computational domain \( \Omega \subset \mathbb{R}^d \), conforming finite element functions are defined locally for the elements \( K \in T_h \) and composed in such a way that the resulting globally defined function belongs to the underlying function space \( V \).

**Definition 3.9 Local trial functions and degrees of freedom**

Let \( T_h \) be a triangulation of \( \Omega \subset \mathbb{R}^d \) and \( K \in T_h \). Assume that \( P_K \) is a linear space of functions \( p : K \to \mathbb{R} \) with \( P_K \subset H^1(K) \) and \( \dim P_K = n_K \). The elements of \( P_K \) are called local trial functions.

Let \( \ell_i \in P_K, 1 \leq i \leq n_K \) be bounded linear functionals \( \ell_i : P_K \to \mathbb{R} \), \( 1 \leq i \leq n_K \), and consider
\[
(3.14) \quad \Sigma_K := \{ \ell_i(p) \mid p \in P_K, \ 1 \leq i \leq n_K \}.
\]

The elements of \( \Sigma_K \) are called degrees of freedom.
**Definition 3.10 Finite elements and unisolvence**

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subseteq \mathbb{R}^d$, $K \in \mathcal{T}_h$ and $P_K, \Sigma_K$ as in Definition 3.9. Then, the triple $(K, P_K, \Sigma_K)$ is called a **finite element**. A finite element $(K, P_K, \Sigma_K)$ is said to be **unisolvent**, if any $p \in P_K$ is uniquely determined by its degrees of freedom in $\Sigma_K$.

**Definition 3.11 Affine equivalence of finite elements**

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subseteq \mathbb{R}^d$, $K \in \mathcal{T}_h$ and $P_K, \Sigma_K$ as in Definition 3.9. Let further $(\hat{K}, \hat{P}_K, \hat{\Sigma}_K)$ be a reference element. Then, the finite elements $(K, P_K, \Sigma_K)$, $K \in \mathcal{T}_h$, are said to be **affine equivalent** to the reference element $(\hat{K}, \hat{P}_K, \hat{\Sigma}_K)$, if there exists an invertible affine mapping $F_K : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $K \in \mathcal{T}_h$

\[
K = F_K(\hat{K}),
\]

\[
P_K = \{ p : K \to \mathbb{R} \mid p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}_K \},
\]

\[
\Sigma_K = \{ \ell_i : P_K \to \mathbb{R} \mid \ell_i = \hat{\ell}_i \circ F_K^{-1}, \hat{\ell}_i \in \hat{\Sigma}_K, 1 \leq i \leq n_K \}.
\]

**Definition 3.12 Finite element space**

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subseteq \mathbb{R}^d$ and $P_K, K \in \mathcal{T}_h$ as in Definition 3.9. Then

\[
V_h := \{ v_h : \overline{\Omega} \to \mathbb{R} \mid v_h|_K \in P_K, K \in \mathcal{T}_h \}
\]

is called a **finite element space**.

**Theorem 3.1 $H^1$-conformity of finite elements**

Let $V_h$ be a finite element space and assume that

\[
P_K \subset H^1(K), \quad K \in \mathcal{T}_h,
\]

\[
V_h \subset C^0(\Omega).
\]

Then

\[
V_h \subset H^1(\Omega).
\]

**Proof.** Let $v_h \in V_h$. Obviously, $v_h \in L^2(\Omega)$. We have to show that $v_h$ admits weak first derivatives $w_h^\alpha \in L^2(\Omega), |\alpha| = 1$, i.e.,

\[
\int_{\Omega} v_h \, D^\alpha z \, dx = (-1)^{|\alpha|} \int_{\Omega} w_h^\alpha \, z \, dx, \quad z \in C^\infty_0(\Omega).
\]
Since $v_h|_{K} \in H^1(K)$, we may apply Green’s theorem elementwise and obtain

\[
\int_{\Omega} v_h D^\alpha z \, dx = \sum_{K \in \mathcal{T}_h} \int_{K} v_h D^\alpha z \, dx = \]

\[
= - \sum_{K \in \mathcal{T}_h} \int_{K} D^\alpha v_h \, z \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} v_h n_\alpha D^\alpha z \, d\sigma = \]

\[
= - \sum_{K \in \mathcal{T}_h} \int_{K} D^\alpha v_h \, z \, dx + \sum_{F \in \mathcal{F}_h(\Omega)} \int_{F} \left[ v_h \right] n_\alpha D^\alpha z \, d\sigma ,
\]

where $[v_h]$ denotes the jump $[v_h] := v_h|_{K_1} - v_h|_{K_2}$ across $F = K_1 \cap K_2, K_i \in \mathcal{T}_h, 1 \leq i \leq 2$. But $v_h \in C^0(\Omega)$, and hence, $[v_h] = 0$ in (3.23) which proves (3.22) with $u_h^0|_{K} := D^\alpha v_h|_{K}, K \in \mathcal{T}_h$. □

**Corollary 3.1** $H^1_0$-conformity of finite elements

Let $V_h$ be a finite element space and assume that

\[
P_K \subset H^1(K) , \quad K \in \mathcal{T}_h ,
\]

\[
V_h \subset C^0(\Omega) .
\]

Then

\[
V_h \subset H^1_0(\Omega) .
\]

**Proof.** The proof is left as an exercise. □

### 3.4 Lagrangian finite elements of type $(k)$

**Definition 3.13 Polynomials of degree $k$**

Let $K$ be a d-simplex. For $k \geq 0$, we define $P_k(K)$ as the linear space of all polynomials of degree $\leq k$ on $K$, i.e., $p \in P_k(K)$, if

\[
p(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha , \quad a_\alpha \in \mathbb{R} , \quad |\alpha| \leq k ,
\]

where

\[
x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i} , \quad \alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d , \quad |\alpha| := \sum_{i=1}^{d} \alpha_i .
\]

We note that

\[
\dim P_k(K) = \binom{k+d}{d} = \frac{(k+d)!}{d! \, k!} .
\]
Definition 3.14 Central lattice of order \( k \) of \( K \)
Let \( K \) be a d-simplex with vertices \( a_i, 1 \leq i \leq d+1 \), and \( k \in \mathbb{N}_0 \). The set
\[
L_k(K) := \{ x = \sum_{i=1}^{d+1} \lambda_i a_i \mid \lambda_i \in \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}, \sum_{i=1}^{d+1} \lambda_i = 1 \}
\]
is called the central lattice of order \( k \) of \( K \). We have
\[
\text{card } L_k(K) = \binom{k+d}{d}.
\]

Definition 3.15 Nodal points
The elements of the central lattice of order \( k \) are called nodal points. For \( D \subseteq \overline{\Omega} \), we refer to \( N_h(D) \) as the set of nodal points in \( D \).

Definition 3.16 Lagrangian finite element of type \( k \)
Let \( T_h \) be a triangulation of \( \Omega \subseteq \mathbb{R}^d \). Then, the triple \( (K, P_k(K), L_k(K)) \), \( K \in T_h \) is called a Lagrangian finite element of type \( (k) \).

Lemma 3.4 Unisolvence of Lagrangian FEs of type \( (k) \)
A Lagrangian finite element of type \( (k) \) is unisolvent.

Proof. The proof is left as an exercise. \( \square \)

Lemma 3.5 Affine equivalence of Lagrangian FEs of type \( (k) \)
Let \( e_i, 1 \leq i \leq d \), be the unit vectors in \( \mathbb{R}^d \) with respect to the Cartesian coordinate system and
\[
\hat{K} := \text{conv}(0, e_1, \ldots, e_d), \quad \hat{P}_k(K) := P_k(\hat{K}), \quad \hat{\Sigma}_K := \{ \hat{p}(\hat{x}) \mid \hat{x} \in L_k(\hat{K}), \hat{p} \in \hat{P}_k(K) \}.
\]
The Lagrangian finite elements of type \( k \) are affine equivalent to the reference element \( (\hat{K}, \hat{P}_k, \hat{\Sigma}_K) \). In particular, \( \hat{K} \) is called the reference \( d \)-simplex.

Proof. Let \( F_K : \mathbb{R}^d \to \mathbb{R}^d \) be the invertible affine mapping with \( K = F_K(\hat{K}) \). Then, \( P_k(K) = \{ p = \hat{p} \circ F_K^{-1} \mid \hat{p} \in P_k(\hat{K}) \} \). Moreover, since \( x \in L_k(K) \iff x = F_K(\hat{x}), \hat{x} \in L_k(\hat{K}) \), (3.17) is easily verified. \( \square \)
Fig. 3.3.a: $P_K = P_1(K)$ (d = 2)  Fig. 3.3.b: $P_K = P_1(K)$ (d = 3)

Fig. 3.4.a: $P_K = P_2(K)$ (d = 2)  Fig. 3.4.b: $P_K = P_2(K)$ (d = 3)

Fig. 3.5.a: $P_K = P_3(K)$ (d = 2)  Fig. 3.5.b: $P_K = P_3(K)$ (d = 3)
Figures 3.3, 3.4, and 3.5 illustrate Lagrangian finite elements of type \((k)\) for \(k = 1, 2, 3\) in two and in three dimensions. Note that a Lagrangian finite element of type \((1)\) for \(d = 2\) is called Courant’s triangle (cf. Fig. 3.3a).

**Definition 3.17** Lagrangian finite element space

The finite element space \(V_h\) composed by Lagrangian finite elements of type \((k)\) is called Lagrangian finite element space and denoted by \(S_k(\Omega, T_h)\).

In order to ensure conformity of \(S_k(\Omega, T_h)\), we have to require that the simplicial triangulation \(T_h\) is geometrically conforming.

**Definition 3.18** Geometrically conforming simplicial triangulation

A simplicial triangulation \(T_h\) of the computational domain \(\Omega \subset \mathbb{R}^d\) is called geometrically conforming, if there holds

\[(T_h) 5\quad \text{The intersection of two different elements of the triangulation is either empty, or consists of a common face, or a common edge, or a common vertex.}\]

**Lemma 3.6** Conformity of Lagrangian finite element spaces

Let \(T_h\) be a geometrically conforming simplicial triangulation of the computational domain \(\Omega \subset \mathbb{R}^d\). Then, there holds

\[(3.30)\quad S_k(\Omega, T_h) = \{ v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in P_k(K), K \in T_h \} \subset H^1(\Omega).\]

**Proof.** Since \(P_k(K) \subset H^1(\Omega), K \in T_h\), in view of Theorem 3.1 we have only to show that \(S_k(\Omega, T_h) \subset C^0(\Omega)\). We give the proof exemplarily in case \(d = 2\) and \(k = 2\):

Let \(K_i \in T_h, 1 \leq i \leq 2\), be two adjacent elements such that \(E = K_1 \cap K_2 \in E_h(\Omega)\) with nodal points \(a_j \in N_h(E), 1 \leq j \leq 3\). Let further \(p_i \in P_2(K_i), 1 \leq i \leq 2\). Then, \(p_i|_E \in P_2(E)\). Since \(p_1(a_j) = p_2(a_j), 1 \leq j \leq 3\), we conclude that \(p_1|_E \equiv p_2|_E\). \(\square\)

**Corollary 3.2** Conformity of Lagrangian finite element spaces

Let \(S_{k,0}(\Omega, T_h) := \{ v_h \in S_k(\Omega, T_h) \mid v_h|_{\partial K \cap \partial \Omega} = 0, K \in T_h, K \cap \partial \Omega \neq \emptyset \}\). Then, under the same assumptions as in Lemma 3.5 there holds

\[(3.31)\quad S_{k,0}(\Omega, T_h) \subset H^1_0(\Omega).\]

**Proof.** The proof is left as an exercise. \(\square\)
Definition 3.19  Nodal basis functions
Let $N_h(\Omega) = \{x_j | 1 \leq j \leq n_h\}$. The function $\varphi_i \in S_k(\Omega, T_h)$ given by
\begin{equation}
\varphi_i(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n_h,
\end{equation}
is called nodal basis function with supporting point $x_i \in N_h(\Omega)$. The set of these functions constitutes a basis of $S_k(\Omega, T_h)$.

Lemma 3.7  Representation by barycentric coordinates
Let $K$ be a $d$-simplex with vertices $a_i, 1 \leq i \leq d + 1$ and set
\begin{align}
a_{ij} &:= \frac{1}{2} (a_i + a_j), \quad 1 \leq i < j \leq d + 1, \\
a_{iij} &:= \frac{1}{3} (2a_i + a_j), \quad 1 \leq i \neq j \leq d + 1, \\
a_{ijk} &:= \frac{1}{3} (a_i + a_j + a_k), \quad 1 \leq i < j < k \leq d + 1.
\end{align}
Let $\lambda_i, 1 \leq i \leq d$, be the barycentric coordinates. Then there holds:

(i) For $k = 1$, the nodal basis functions $\varphi_i$ associated with the nodal points $a_i, 1 \leq i \leq d + 1$, admit the representation
\begin{equation}
\varphi_i = \lambda_i, \quad 1 \leq i \leq d + 1.
\end{equation}

(ii) For $k = 2$, the nodal basis functions
- $\varphi_i$ associated with the nodal points $a_i, 1 \leq i \leq d + 1$, admit the representation
\begin{equation}
\varphi_i = \lambda_i (2\lambda_i - 1), \quad 1 \leq i \leq d + 1.
\end{equation}
- $\varphi_{ij}$ associated with the nodal points $a_{ij}, 1 \leq i < j \leq d + 1$, admit the representation
\begin{equation}
\varphi_{ij} = 4 \lambda_i \lambda_j, \quad 1 \leq i < j \leq d + 1.
\end{equation}

(iii) For $k = 3$, the nodal basis functions
- $\varphi_i$ associated with the nodal points $a_i, 1 \leq i \leq d + 1$, admit the representation
\begin{equation}
\varphi_i = \frac{1}{2} \lambda_i (3\lambda_i - 1) (3\lambda_i - 2), \quad 1 \leq i \leq d + 1.
\end{equation}
- $\varphi_{iij}$ associated with the nodal points $a_{iij}, 1 \leq i \neq j \leq d + 1$, admit the representation
\begin{equation}
\varphi_{iij} = \frac{1}{2} 9 \lambda_i \lambda_j (3\lambda_i - 1), \quad 1 \leq i \leq d + 1.
\end{equation}
\( \varphi_{ijk} \) associated with the nodal points \( a_{ijk} \), \( 1 \leq i < j < k \leq d+1 \), admit the representation

\[
\varphi_{ijk} = 27 \lambda_i \lambda_j \lambda_k, \quad 1 \leq i < j < k \leq d+1.
\]

**Proof.** The proof is left as an exercise. \( \square \)

![Fig. 3.6: Non degenerate triangle with vertices \( a_i \) and edges \( e_i \)](image)

**Lemma 3.8 Geometric characterization**

Let \( K \) be a non degenerate 2-simplex with vertices \( a_i, 1 \leq i \leq 3 \), and edges \( e_i, 1 \leq i \leq 3 \) (cf. Fig. 3.6). Moreover, let \( m_{e_i} \) be the midpoints of the edges and denote by \( n_{e_i} \) the exterior unit normals to \( e_i \). Then, the nodal basis functions \( \varphi_i, 1 \leq i \leq 3 \), associated with the vertices \( a_i \) admit the representation

\[
\varphi_i(x) = \frac{\text{meas}(e_i)}{2\text{meas}(K)} n_{e_i} \cdot (m_{e_i} - x), \quad x = (x_1, x_2)^T.
\]

**Proof.** The proof is left as an exercise. \( \square \)

**3.5 Hermitian finite elements of type (3)**

The name ‘Lagrangian finite elements’ is motivated by polynomial interpolation of Lagrangian type. Indeed, if \( \Pi_K; K \in T_h \), is the associated local interpolation operator, \( \Pi_K v, v \in C^0(K) \), interpolates the function \( p \) in \( x \in L_k(K) \). Another type of polynomial interpolation is **Hermite interpolation**, where both point values and derivatives of a function are interpolated. The counterpart in finite elements are **Hermitian finite elements**. As an example, we consider Hermitian finite elements of type 3.

**Definition 3.20 Hermitian finite elements of type (3)**

Let \( K \) be a non degenerate \( d \)-simplex with vertices \( a_i, 1 \leq i \leq d+1 \), and \( a_{ijk} := \frac{1}{3}(a_i + a_j + a_k), 1 \leq i < j < k \leq d+1 \). Moreover, let

\[
P_K := P_3(K),
\]
\( \Sigma_K := \{ p(a_i), 1 \leq i \leq d+1, \\
p(a_{ijk}), 1 \leq i < j < k \leq d+1, \\
\frac{\partial p}{\partial x_j}(a_i), 1 \leq i \leq d+1, 1 \leq j \leq d \} \).

Then, \( (K, P_K, \Sigma_K) \) is called a Hermitian finite element of type (3).

**Remark 3.1 Equivalent specification of the DOFs**

We note that in the definition of the degrees of freedom of the Hermitian finite element of type (3) the partial derivatives \( \frac{\partial p}{\partial x_j}(a_i), 1 \leq i \leq d+1, 1 \leq j \leq d \), can be replaced by the directional derivatives \( Dp(a_i)(a_j - a_i), 1 \leq i \leq d+1, 1 \leq j \leq d \), i.e., instead of \( \Sigma_K \) we have

\( \Sigma'_K := \{ p(a_i), 1 \leq i \leq d+1, \\
p(a_{ijk}), 1 \leq i < j < k \leq d+1, \\
Dp(a_i)(a_j - a_i), 1 \leq i \leq d+1, 1 \leq j \leq d \} \).

**Lemma 3.9 Unisolvence of Hermitian finite elements**

A Hermitian finite element of type (3) is unisolvent.

**Proof.** We give the proof exemplarily in case \( d = 3 \). Let \( p \in P_3(K) \).

It suffices to show that

\begin{align*}
(3.44) & \quad p(a_i) = 0, \ 1 \leq i \leq 4, \\
(3.45) & \quad p(a_{ijk}) = 0, \ 1 \leq i < j < k \leq 4, \\
(3.46) & \quad Dp(a_i)(a_j - a_i) = 0, \ 1 \leq i \leq 4, 1 \leq j \leq 3
\end{align*}

implies \( p \equiv 0 \).

Let \( F_{ijk}, 1 \leq i < j < k \leq 4 \), be the face with vertices \( a_i, a_j, a_k \) and let \( e_{ij}, e_{ik}, e_{jk} \) be the corresponding edges. Then

\[
p|_{e_{ij}} \in P_3(e_{ij}), \quad p|_{e_{ik}} \in P_3(e_{ik}), \quad p|_{e_{jk}} \in P_3(e_{jk}) .
\]

Since the **Hermite interpolation polynomial** interpolating

\[
p(a_i), \ p(a_j), \ Dp(a_i)(a_j - a_i), \ Dp(a_j)(a_i - a_j)
\]

is uniquely determined, (3.44) and (3.46) imply \( p|_{e_{ij}} \equiv 0 \). Likewise, we deduce \( p|_{e_{ik}} \equiv 0 \) and \( p|_{e_{jk}} \equiv 0 \), and hence

\[
p|_{\partial F_{ijk}} \equiv 0 .
\]

Consequently, \( p \) is of the form

\[
p = \alpha \lambda_i \lambda_j \lambda_k , \quad \alpha \in \mathbb{R} .
\]
Then, (3.45) implies \( \alpha = 0 \), i.e.,
\[
p_{|_{F_{ijk}}} \equiv 0, \quad 1 \leq i < j < k \leq 4 \quad \Rightarrow \quad p \equiv 0. \quad \square
\]

Figure 3.6: Hermitian finite element of type (3)

Figure 3.6 contains a schematic representation of Hermitian finite elements of type (3). The degrees of freedom, as given by point values, are marked by a dot, whereas the degrees of freedom, associated with partial resp. directional derivatives, are indicated by a circle.

**Lemma 3.10 Affine equivalence of Hermitian finite elements**

Let \( \hat{K} \) be the reference \( d \)-simplex and
\[
\hat{P}_{\hat{K}} := P_3(\hat{K}),
\]
\[
\hat{\Sigma}_{\hat{K}} := \{ \hat{p}(\hat{a}_i), \ 1 \leq i \leq d+1, \ \hat{p}(\hat{a}_{ijk}), \ 1 \leq i < j < k \leq d+1, \ \frac{\partial \hat{p}}{\partial \hat{x}_j}(\hat{a}_i), \ 1 \leq i \leq d+1, \ 1 \leq j \leq d \}.
\]
The Hermitian finite element \( K, P_K, \Sigma_K \) of type (3) is affine equivalent to the reference element \( \hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}} \) of type (3).

**Proof.** We only have to show how the partial derivatives are transformed. Let \( F_K : \mathbb{R}^d \to \mathbb{R}^d \) be the invertible affine mapping \( F_K(\hat{x}) = B_K \hat{x} + b_K \) such that \( K = F_K(\hat{K}) \). Then, for \( \hat{p} \in P_3(\hat{K}) \) and \( p = \hat{p} \circ F_K^{-1} \),
we have
\[ \frac{\partial p}{\partial x_i}(a_j) = \sum_{k=1}^{d} \frac{\partial \hat{p}}{\partial \hat{x}_k} \frac{\partial (F^{-1}_K(a_j))_k}{\partial x_i}, \]
whence
\[ Dp(a_j) = B^{-1}_K \hat{D}\hat{p}(\hat{a}_j). \]

**Definition 3.21 Hermitian finite element space**

For a simplicial triangulation $T_h$ of the polyhedral domain $\Omega \subset \mathbb{R}^d$, the finite element space $V_h$ composed by Hermitian finite elements of type (3) is called a **Hermitian finite element space**. It will be denoted by $H_3(\Omega; T_h)$.

**Lemma 3.11 Conformity of Hermitian finite element spaces**

Let $T_h$ be a geometrically conforming simplicial triangulation of $\Omega \subset \mathbb{R}^d$. Then, the the Hermitian finite element space $H_3(\Omega; T_h)$ is **conforming**, i.e.,
\[ H_3(\Omega; T_h) \subset H^1(\Omega) . \]

**Proof.** The proof is left as an exercise.

**Definition 3.22 Basis functions of Hermitian finite element spaces**

We restrict ourselves to the case $d = 2$ and assume that $H_3(\Omega; T_h)$ is the Hermitian finite element space with respect to a geometrically conforming simplicial triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$.

We denote by $b_{\ell}, 1 \leq \ell \leq r_h$, the nodal points that are vertices of triangles $T \in T_h$ and by $b_{\ell}, r_h + 1 \leq \ell \leq s_h$, those that are centers of gravity of $T \in T_h$. Then, an appropriate basis
\[ \varphi_k, 1 \leq k \leq s_h \ , \ \varphi^{(\nu)}_k, 1 \leq k \leq r_h , \ 1 \leq \nu \leq 2 , \]
is given by
\[ \varphi_k(b_{\ell}) = \delta_{k\ell} , 1 \leq k, \ell \leq s_h , \]
\[ \frac{\partial \varphi_k}{\partial x_{\nu}}(b_{\ell}) = 0 , 1 \leq k \leq s_h , 1 \leq \ell \leq r_h , 1 \leq \nu \leq 2 , \]
\[ \varphi^{(1)}_k(b_{\ell}) = 0 , 1 \leq k \leq r_h , 1 \leq \ell \leq s_h , \]
\[ \frac{\partial \varphi^{(1)}_k}{\partial x_1}(b_{\ell}) = \delta_{k\ell} , \frac{\partial \varphi^{(1)}_k}{\partial x_2} (b_{\ell}) = 0 , 1 \leq k, \ell \leq r_h , \]
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(3.51) \[ \varphi_k^{(2)}(b_{\ell}) = 0, \ 1 \leq k \leq r_h, \ 1 \leq \ell \leq s_h, \]
\[ \frac{\partial \varphi_k^{(2)}}{\partial x_1}(b_{\ell}) = 0, \ \frac{\partial \varphi_k^{(2)}}{\partial x_2}(b_{\ell}) = \delta_{k\ell}, \ 1 \leq k, \ell \leq r_h. \]

3.6 Lagrangian finite elements of type \([k]\)

Let \( K := \prod_{i=1}^{d} [c_i, d_i] \subset \mathbb{R}^d \) be a d-rectangle. For \( k \in \mathbb{N}_0 \) we denote by \( Q_k(K) \) the linear space of all polynomials that are of degree \( \leq k \) in each of the \( d \) variables \( x_i, 1 \leq i \leq d \), i.e., \( p \in Q_k(K) \) is of the form

(3.52) \[ p(x) = \sum_{\alpha_i \leq k} a_{\alpha_1...\alpha_d} x_1^{\alpha_1} x_2^{\alpha_2} ... x_d^{\alpha_d}, \]

from which we easily deduce

(3.53) \[ \dim Q_k(K) = (k + 1)^d. \]

For the \( d \)-rectangle \( K \) we consider the point-set

(3.54) \[ L_{[k]}(K) := \{ x = (x_{i_1}, ..., x_{i_d})^T \mid x_{i_{\ell}} := c_{\ell} + \frac{i_{\ell}}{k}(d_{\ell} - c_{\ell}), \ i_{\ell} \in \{0, 1, ..., k\}, \ 1 \leq \ell \leq d \} \]

and define

(3.55) \[ \Sigma_K := \{ p(x) \mid x \in L_{[k]}(K) \}, \ p \in Q_k(K). \]

Then there holds:

**Definition 3.22 Lagrangian finite element of type \([k]\)**

The element \((K, Q_k(K), \Sigma_K)\) is called a **Lagrangian finite element of type \([k]\)** and will be denoted by \( S_{[k]}(K) \). The points \( x \in L_{[k]}(K) \) are referred to as **nodal points**.

**Lemma 3.12 Lagrangian finite element of type \([k]\) as a tensor product finite element**

The Lagrangian finite element of type \([k]\) is a tensor product finite element based on **tensor product Lagrangian type polynomial interpolation**:

The polynomial \( p \in Q_k(K) \), interpolating in \( x \in L_{[k]}(K) \), has the representation

(3.56) \[ p(x) = \sum_{y \in L_{[k]}(K)} L(x) \ p(y). \]
Here, the Lagrangian fundamental polynomial $L \in Q_k(K)$ is given by

$$L(x) = \prod_{\ell=1}^{d} L_{i_{\ell}}(x)$$

in terms of the one-dimensional Lagrangian fundamental polynomials

$$L_{i_{\ell}}(x) := \prod_{i_{\ell} \neq i'_{\ell} = 0}^{k} \frac{x - x_{i'_{\ell}}}{x_{i_{\ell}} - x_{i'_{\ell}}} , \quad 1 \leq \ell \leq d .$$

**Proof.** Let $p \in Q_k(K)$ and consider an arbitrary edge $e_j \subset K, j \in \{1, d\}$ as given by

$$e_j := [c_j, d_j] \times \prod_{j \neq \ell = 1}^{d} \{g_{\ell}\} , \quad g_{\ell} \in \{c_{\ell}, d_{\ell}\} .$$

Then, $p|_{e_j} \in P_k(e_j)$ is uniquely determined by its values in $x_{i_j} := c_j + \frac{i_{\ell}}{K}(d_j - c_j) \in e_j, i_j \in \{0, 1, ..., k\}$, and has the Lagrangian representation

$$p|_{e_j}(x) = \sum_{i_j=0}^{k} L_{i_{\ell}}(x) p|_{e_j}(x_{i_j}) , \quad x \in e_j .$$

**Lemma 3.13 Unisolvence**

The Lagrangian finite element of type $([k])$ is unisolvent.

**Proof.** The proof is an immediate consequence of Lemma 3.12. □

**Lemma 3.14 Affine equivalence of Lagrangian FEs of type $[k]$**

Let $\hat{K} := [0, 1]^d$ be the unit cube in $\mathbb{R}^d$ and

$$\hat{P}_K := Q_k(\hat{K}) , \quad k \in \mathbb{N} ,$$

$$\hat{\Sigma}_K := \{\hat{p}(\hat{x}) \mid \hat{x} \in L_{[k]}(\hat{K}) , \hat{p} \in \hat{P}_K\} .$$

The Lagrangian finite elements of type $[k]$ are affine equivalent to the reference element $(\hat{K}, \hat{P}_K, \hat{\Sigma}_K)$. In particular, $\hat{K}$ is called the reference $d$-rectangle.

**Proof.** The proof is left as an exercise. □
For $k = 1$ and $k = 2$, Figures 3.7 and 3.8 contain illustrations of Lagrangian finite elements of type $[k]$. 

**Definition 3.23** Lagrangian finite element space

Let $\Omega \subset \mathbb{R}^d$ be given as the union of a finite number of $d$-rectangles and let $T_h$ be a rectangular triangulation of $\Omega$. The finite element space $V_h$ composed by Lagrangian finite elements of type $[k]$ is called Lagrangian finite element space and denoted by $S_{[k]}(\Omega, T_h)$.

**Lemma 3.15** Conformity of Lagrangian finite element spaces

Let $T_h$ be a geometrically conforming rectangular triangulation of the computational domain $\Omega$. Then, there holds

\[(3.61) \quad S_{[k]}(\Omega, T_h) = \{v_h \in C^0(\Omega) \mid v_h|_K \in Q_k(K), K \in T_h\} \subset H^1(\Omega).\]

**Proof.** The proof is left as an exercise. \qed
Corollary 3.3  Conformity of Lagrangian finite element spaces

Let \( S_{[k],0}(\Omega, T_h) := \{ v_h \in S_{[k]}(\Omega, T_h) \mid v_h|_{\partial K \cap \Omega} = 0, K \in T_h, K \cap \partial \Omega \neq \emptyset \} \). Then, under the same assumptions as in Lemma 3.15 there holds

\[
S_{[k],0}(\Omega, T_h) \subset H^1_0(\Omega).
\]

**Proof.** The proof is left as an exercise. \( \square \)

**Definition 3.24  Nodal basis functions**

Let \( \mathcal{N}_h(\Omega) = \{ x \in L_{[k]}(K) \mid K \in T_h \} \) and suppose \( \text{card}(\mathcal{N}_h(\Omega)) = n_h \).

The function \( \varphi_i \in S_{[k]}(\Omega, T_h) \) given by

\[
\varphi_i(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n_h ,
\]

is called **nodal basis function with supporting point** \( x_i \in \mathcal{N}_h(\Omega) \).

The set of these functions constitutes a basis of \( S_{[k]}(\Omega, T_h) \).