5 Shooting methods for boundary value problems

5.1 Foundations

Definition 5.1 Two-point boundary value problem

Let $I := [a, b] \subset \mathbb{R}$, $D \subset \mathbb{R}^n$, and $f : I \times D \to \mathbb{R}^n$ as well as $r : D \times D \to \mathbb{R}^n$. Then,

\begin{align*}
(*)_1 & \quad y'(x) = f(x, y(x)) \quad x \in I, \\
(*)_2 & \quad r(y(a), y(b)) = 0
\end{align*}

is called a two-point boundary value problem for the system $(*)_1$ of first order ordinary differential equations.

Existence and uniqueness results

Example 5.2 Scalar second order differential equation

Consider the scalar second order differential equation

\begin{align*}
(*)_1 & \quad w''(x) + w(x) = 0 \quad , \quad x \in [0, b] \quad b = \pi/2 \quad \text{or} \quad b = \pi
\end{align*}

with boundary conditions

\begin{align*}
(i) & \quad w(\pi/2) = 1, \\
(ii) & \quad w(\pi) = 0, \\
(iii) & \quad w(\pi) = 1.
\end{align*}
Setting

\[ y_1(x) := w(x), \quad y_2(x) := w'(x), \]

the system (\(*_1\), \(*_2\)) can be reformulated as (\(*_1\), \(*_2\)):

\[
\begin{align*}
(*_1) & \quad \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} y_2(x) \\ -y_1(x) \end{pmatrix}, \quad x \in [0, b], \\
(*_2) & \quad r(y(0), y(b)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(b) \\ y_2(b) \end{pmatrix} - \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}
\]

where \( d = 1 \) for (\(*_2\) \(i\), \(iii\)) and \( d = 0 \) for (\(*_2\) \(ii\)).

Fundamental system of (\(*_1\)):

\[
\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}, \quad \begin{pmatrix} -\cos x \\ \sin x \end{pmatrix}.
\]

\[ \Rightarrow \quad \text{General solution:} \]

\[ w(x) = c_1 \sin x + c_2 \cos x. \]
(i) \( w(0) = 0 \), \( w(\pi/2) = 1 \)

Eindeutige Lösung: \( w(x) = \sin x \)

(ii) \( w(0) = 0 \), \( w(\pi) = 0 \)

Manifold of solutions: \( w(x) = c_1 \sin x \), \( c_1 \in \mathbb{R} \)

(iii) \( w(0) = 0 \), \( w(\pi) = 1 \)

No solution
Local uniqueness: Functional analytic approach

Formulation of $(*)_1, (*)_2$ as operator equation:

Compute $y^* \in C^1(I)$ such that

$$Ty^* = 0,$$
where $T : C^1(I) \to C^1(I) \times \mathbb{R}^n$,

$$Ty := \begin{cases} y(x) - y(a) - \int_a^x f(s, y(s)) \, ds, & x \in I, \\ r(y(a), y(b)). \end{cases}$$

The solution $y^* \in C^1(I)$ is locally unique, if there holds:

There exists a neighborhood $U(y^*) \subset C^1(I)$ such that $T$ is injective in $U(y^*)$. Equivalent condition:

$$(\dagger) \quad T'y^* \delta y = 0 \implies \delta y = 0,$$

where $T'(y^*)$ denotes the Fréchet derivative of $T$ in $y^*$:

$$T'y^* \delta y = \begin{cases} \delta y(x) - \delta y(a) - \int_a^x f_y(s, y^*(s)) \, \delta y(s) \, ds, \\ \frac{\partial r}{\partial y_a}(y^*(a), y^*(b)) \, \delta y(a) + \frac{\partial r}{\partial y_b}(y^*(a), y^*(b)) \, \delta y(b). \end{cases}$$

$$= A^* + B^*$$
Equivalence of (†) with the linear boundary value problem:

\[(‡)_1 \quad (\delta y)'(x) = f_y(x, y^*(x)) \, (\delta y)(x) \quad , \quad x \in I \, , \]

\[(‡)_2 \quad A^*(\delta y)(a) + B^*(\delta y)(b) = 0 \, . \]

Let \( W^*(x_0, x) \, , \ x_0 \in I \), be the **Wronskian matrix** with respect to (‡)_1, i.e.,

\[
\frac{dW^*}{dx}(x_0, x) = f_y(x, y^*(x)) \, W^*(x_0, x) \, , \ x_0 \in I \, , \\
W^*(x_0, x_0) = I \, . 
\]

Set:

\[
(\delta y)(x) = W^*(a, x) \, (\delta y)(a) \, .
\]

Since \( W^*(a, x) \) is non-singular, there holds:

\[
(\delta y)(a) = 0 \iff (\delta y)(x) = 0 \, , \ x \in I \, .
\]

In view of \((\delta y)(b) = W^*(a, b)(\delta y)(a)\) and observing (‡)_2, there holds:

\[
\left[ A^* + B^* \, W^*(a, b) \right] \, (\delta y)(a) = 0 \, .
\]

Hence, we have \((\delta y)(a) = 0\) if and only if \( \det[A^* + B^* \, W^*(a, b)] \neq 0 \, . \)
Definition 5.3  Sensitivity matrix
The matrix given by
\[ E^*(x) := A^* W^*(a, x) + B^* W^*(b, x) \]
is called the sensitivity matrix in \( x \in I \).

Lemma 5.4  Properties of the sensitivity matrix
There holds:
\[ E^*(x) = E^*(a) W^*(a, x), \quad x \in I. \]

Consequently, \( E^*(x) \) is non singular for all \( x \in I \) if and only if \( E^*(x) \) is non singular for some \( x \in I \).

Proof:
\[ E^*(a) W^*(a, x) = (A^* + B^* W^*(b, a)) W^*(a, x) = A^* W^*(a, x) + B^* W^*(b, a) W^*(a, x) = E^*(x). \]

Theorem 5.5  Local uniqueness
A solution \( y^* \in C^1(I) \) of the boundary value problem \((*)_1, (*)_2\) is locally unique, if the sensitivity matrix \( E^*(x) \) is non singular for some \( x \in I \).
Example 5.2 (revisited)

\[ A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

(i) Boundary conditions: \( w(0) = 0 \), \( w(\pi/2) = 1 \)

\[ W^*(0, x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}, \quad W^*(\pi/2, x) = \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}, \]

\[ E^*(x) = A^* W^*(0, x) + B^* W^*(\pi/2, x) = \begin{pmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{pmatrix}, \quad \det E^*(x) = -1 \neq 0, \quad x \in [0, \pi/2]. \]

(ii) Boundary conditions: \( w(0) = 0 \), \( w(\pi) = 0 \)

\[ W^*(0, x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}, \quad W^*(\pi, x) = \begin{pmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{pmatrix}, \]

\[ E^*(x) = A^* W^*(0, x) + B^* W^*(\pi, x) = \begin{pmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{pmatrix}, \quad \det E^*(x) = 0, \quad x \in [0, \pi]. \]
Definition 5.6  Condition with respect to perturbations in the boundary data

The quantities given by

\[ \rho_a(x) := \|E^*(x)^{-1} A^*\| , \quad \rho_b(x) := \|E^*(x)^{-1} B^*\| , \]

\[ \rho(x) := \rho_a(x) + \rho_b(x) , \quad \bar{\rho} := \max_{x \in [a, b]} [\rho_a(x) + \rho_b(x)] \]

are called the condition numbers with respect to perturbations in the boundary data.

Theorem 5.7  Condition number estimate (perturbations in the boundary data)

Assume that \( y \in C^1(I) \) is the solution of the boundary value problem \((\ast)_1,(\ast)_2\) and that \( \tilde{y} \in C^1(I) \) is the solution of the perturbed boundary value problem

\[ (\ast)'_1 \quad \tilde{y}'(x) = f(x, \tilde{y}(x)) , \quad x \in I , \]

\[ (\ast)'_2 \quad r(\tilde{y}(a), \tilde{y}(b)) - \delta r = 0 . \]

For \( \delta y_r := \tilde{y} - y \), we have:

\[ \|\delta y_r(x)\|_\infty \leq \rho(x) \max [ \|\delta y_r(a)\|_\infty, \|\delta y_r(b)\|_\infty ] , \]

\[ \max_{x \in [a, b]} \|\delta y_r(x)\|_\infty \leq \bar{\rho} \max [ \|\delta y_r(a)\|_\infty, \|\delta y_r(b)\|_\infty ] . \]
Proof: At first approximation, we have

\[ \delta y_r(x) = (\tilde{y} - y)(x) = f(x, \tilde{y}(x)) - f(x, y(x)) = f_y(x, y(x)) (\tilde{y} - y)(x) , \]
\[ r(\tilde{y}(a), \tilde{y}(b)) - \delta r + r(y(a), y(b)) \doteq \frac{\partial r}{\partial y_a}(y(a), y(b)) (\tilde{y} - y)(a) + \frac{\partial r}{\partial y_b}(y(a), y(b)) (\tilde{y} - y)(b) - \delta r. \]

Hence, at first approximation \( \delta y_r \) satisfies the linear boundary value problem

\[ \begin{align*}
(\delta y_r)'(x) &= f_y(x, y(x)) (\delta y_r)(x) , \quad x \in I , \\
A (\delta y_r)(a) + B (\delta y_r)(b) &= \delta r .
\end{align*} \]

It follows that

\[ (\delta y_r)(x) = W(a, x) (\delta y_r)(a) = W(a, x) E^*(a)^{-1} \delta r = E^*(x)^{-1} \delta r = E^*(x)^{-1} [A^* (\delta y_r)(a) + B^* (\delta y_r)(b)] . \]

Remark 5.8 About the 'sensitivity matrix'

The condition number estimate in Theorem 5.7 shows that we may expect problems, if \( E^*(x) \) is almost singular. This observation motivates the notion 'sensitivity matrix'.
Consider the linear boundary value problem

\begin{align*}
&(\dagger)_1 \quad (\delta y_f)'(x) = f_y(x,y(x)) \,(\delta y_f)(x) + \delta f(x) \quad , \quad x \in I, \\
&(\dagger)_2 \quad A^* \,(\delta y_f)(a) + B^* \,(\delta y_f)(b) = 0
\end{align*}

with the Wronski matrix $W^*(x_0,x)$ and the sensitivity matrix $E^*(x)$.

**Definition 5.9  Green’s function**

The function given by

$$G(x,s) := \begin{cases} 
E^*(x)^{-1} \, A^* \, W^*(a,s) , & a \leq s \leq x , \\
- E^*(x)^{-1} \, A^* \, W^*(b,s) , & x \leq s \leq b
\end{cases}$$

is called **Green’s function** of the linear boundary value problem $(\dagger)_1, (\dagger)_2$.

**Theorem 5.10  Solution in terms of Green’s function**

If the sensitivity matrix $E^*(x)$ is non singular, the unique solution of the linear boundary value problem $(\dagger)_1, (\dagger)_2$ is given by

$$(\delta y_f)(x) = \int_{a}^{b} G(x,s) \, \delta f(s) \, ds .$$
Definition 5.11  Condition number with respect to perturbations of the right-hand side

Let $G(\cdot, \cdot)$ be Green’s function of the linear boundary value problem $(\dagger)_1, (\dagger)_2$. Then, the quantity

$$\kappa_f := (b - a) \max_{x, s \in [a, b]} \|G(x, s)\|$$

is called the condition number with respect to perturbations of the right-hand side.

Theorem 5.12  Condition number estimate (perturbations of the right-hand side)

Assume that $y \in C^1(I)$ is the solution of the boundary value problem $(\ast)_1, (\ast)_2$ and that $\tilde{y} \in C^1(I)$ is the solution of the perturbed boundary value problem

$$\tilde{y}'(x) = f(x, \tilde{y}(x)) + \delta f(x) \quad , \quad x \in I,$$

$$r(\tilde{y}(a), \tilde{y}(b)) = 0.$$

For $\delta y_f := \tilde{y} - y$ we have at first approximation

$$\max_{x \in [a, b]} \|\delta y_f(x)\|_{\infty} \leq \kappa_f \max_{x \in [a, b]} \|\delta f(x)\|_{\infty}.$$

Proof: At first approximation, $\delta y_f$ satisfies the linear boundary value problem $(\ddagger)_1, (\ddagger)_2$. 

5.2 Simple shooting method

Example 5.12 Boundary value problem for 2nd order DE

(\star)_1 \quad y''(x) = f(x, y(x), y'(x)) \quad x \in [a, b],

(\star)_2 \quad y(a) = \alpha, \ y(b) = \beta.

Idea: Compute the initial slope $s$ of the initial value problem

(\circ)_1 \quad y''(x) = f(x, y(x), y'(x)) \quad x \in [a, b],

(\circ)_2 \quad y(a) = \alpha, \ y'(a) = s

such that

(\circ)_3 \quad y(b) = y(b; s) = \beta,

where $y(\cdot; s)$ is the solution of the initial value problem

(\circ)_1, (\circ)_2 with initial slope $s$.

This means: We are looking for a zero $s \in \mathbb{R}$ of the non-linear mapping

(\bullet) \quad F(s) := y(b; s) - \beta.
For the solution of $F(s^\nu) := Y(b; s) - \beta = 0$ we use Newton's method:

$$s^{(\nu+1)} = s^{(\nu)} - \frac{F(s^{(\nu)})}{F'(s^{(\nu)})} , \quad \nu \geq 0 .$$

(i) The computation of $F(s^\nu)$ requires the solution of the initial value problem

$$y''(x) = f(x, y(x), y'(x)) , \quad x \in I ,$$

$$y(a) = \alpha , \quad y'(a) = s^{(\nu)} .$$

(ii) Computation of $F'(s^{(\nu)})$:

Observe $F'(s) = \frac{\partial}{\partial s} y(b; s)$. The function $z(x; s) := \frac{\partial}{\partial s} y(x; s)$ satisfies the initial value problem

$$z''(x) = f_y(x, y(x), y'(x)) z(x) + f_y'(x, y(x), y'(x)) z'(x) , \quad x \in I ,$$

$$z(a) = 0 , \quad z'(a) = 1 .$$
Two strategies for the realization of Newton’s method:

(i) External numerical differentiation

Replace $F'(s^{(\nu)})$ by the difference quotient

$$\Delta F(s^{(\nu)}) := \frac{F(s^{(\nu)} + \Delta s^{(\nu)}) - F(s^{(\nu)})}{\Delta s^{(\nu)}}$$

Remarks:

(α) The computation of $F(s^{(\nu)} + \Delta s^{(\nu)})$ requires the solution of an initial value problem.

(β) Sensitivity with respect to rounding errors.

(ii) Internal numerical differentiation

For the computation of $F'(s^{(\nu)})$, replace $f_y$ and $f_{y'}$ by

$$\frac{f(x, y + \Delta y, y') - f(x, y, y')}{\Delta y}, \quad \frac{f(x, y, y' + \Delta y') - f(x, y, y')}{\Delta y'}.$$

Remark: Less sensitive with respect to rounding errors.
Simple shooting method for general boundary value problems:

\[ \begin{align*}
(\circ)_1 & \quad y'(x) = f(x, y(x)) , \quad x \in [a, b] , \\
(\circ)_2 & \quad r(y(a), y(b)) = 0 .
\end{align*} \]

Compute \( s \in \mathbb{R}^n \) such that for the solution of the initial value problem

\[ \begin{align*}
(\dagger)_1 & \quad y'(x; s) = f(x, y(x; s)) , \quad x \in [a, b] , \\
(\dagger)_2 & \quad y(a; s) = s
\end{align*} \]

there holds

\[ F(s) := r(y(a; s), y(b; s)) = r(s, y(b; s)) = 0 . \]

Newton’s method:

\[ \begin{align*}
F_s(s^{(\nu)}) \Delta s^{(\nu)} &= - F(s^{(\nu)}) , \\
s^{(\nu+1)} &= s^{(\nu)} + \Delta s^{(\nu)} ,
\end{align*} \]

where

\[ F_s(s) = \frac{\partial r}{\partial y_a}(s, y(b; s)) + \frac{\partial r}{\partial y_b}(s, y(b; s)) \ W_s(b, a) . \]
Here, $W_s(x, a)$ denotes the solution of

$$\frac{d}{dx} W_s(x, a) = f_y(x, y(x; s)) W_s(x, a),$$

$$W_s(a, a) = I.$$

The computation of $A, B$ and $f_y$ is done by numerical differentiation.

Disadvantages of simple shooting:

(i) Moving singularities

Often, it is not possible to compute the trajectory $y(x; s)$ for the entire interval $[a, b]$, since for the solution of the initial value problem $(\dagger)_1, (\dagger)_2$ singularities in $[a, b]$ may occur that depend on $s$: In case $s$ is estimated badly, $y(x; s)$ can be unbounded in $[a, b]$.

Example 5.13 Singularity

$$y''(x) = \lambda \sinh(\lambda y(x)), \quad x \in [0, 1],$$

$$y(0) = 0, \quad y(1) = 1.$$
For $s \to 0$, the initial value problem

$$y''(x) = \lambda \sinh(\lambda y(x)) \quad , \quad x \in [0, 1] ,$$
$$y(0) = 0 \quad , \quad y'(0) = s$$

has a singularity in

$$x_s = \frac{1}{\lambda} \ln\left(\frac{8}{|s|}\right) \quad , \quad (x_s > 1 !) ,$$

i.e., $s$ must be chosen such that

$$|s| \leq 8 \exp(-\lambda) \quad , \quad \text{for instance} \quad |s| \leq 0.05 \quad \text{for} \quad \lambda = 5 .$$
(ii) Stability problems
Even in case of well conditioned boundary value problems, the associated initial value problem can be ill conditioned.

Example 5.14 Stability problem
\[ y''(x) = \lambda^2 y(x), \quad x \in [a, b], \]
\[ y(a) = \alpha, \quad y(b) = \beta. \]

Condition of the boundary value problem
\[ \bar{\rho} \doteq \lambda \quad (\lambda \to \infty). \]

Condition of the associated initial value problem
\[ y''(x) = \lambda^2 y(x), \quad x \in [a, b], \]
\[ y(a) = \alpha, \quad y'(a) = s. \]

\[ \sigma(a, b) = \max_{x \in [a, b]} \|W(x, a)\| \doteq \lambda \exp(\lambda(b - a)) \quad (\lambda \to \infty). \]
5.3 Multiple shooting

Idea: Partition the interval $I := [a, b]$ according to

$$\Delta_m := \{ \ x_1 := a < x_2 < \ldots < x_m := b \ \} \ , \ m > 2$$

and compute $y(x; s_k)$, $x \in [x_k, x_{k+1}]$ for $k = 1, \ldots, m - 1$

as the solution of the initial value problems

$$y'(x) = f(x, y(x)) \ , \ x \in [x_k, x_{k+1}] \ ,$$

$$y(x_k) = s_k \ .$$

Determine $s_1, \ldots, s_{m-1}$ and $s_m$ such that the composite function

$$y(x) := y(x; s_k) \ , \ x \in [x_k, x_{k+1}] \ , \ 1 \leq k \leq m - 1 \ ,$$

$$y(b) := s_m$$

is continuous and satisfies the boundary conditions

$$(*)_1 \ \ F_k(s_k, s_{k+1}) := y(x_{k+1}; s_k) - s_{k+1} = 0 \ , \ 1 \leq k \leq m - 1 \ ,$$

$$(*)_2 \ \ F_m(s_1, s_m) := r(s_1, s_m) = 0 \ .$$

The system $(*)_1, (*)_2$ is called a cyclic linear system in the $n \cdot m$ unknowns

$$s_k = (s_{k,1}, \ldots, s_{k,m})^T \ , \ 1 \leq k \leq m.$$
Newton’s method for the solution of $(\ast)_1, (\ast)_2$

$(\diamond)_1 \quad J(s^{(\nu)}) \Delta s^{(\nu)} = -F(s^{(\nu)}), \quad \nu \geq 0,$

$(\diamond)_2 \quad s^{(\nu+1)} = s^{(\nu)} + \Delta s^{(\nu)},$

where

$$J(s) = \left( \frac{\partial F_k}{\partial s_{\ell}} \right)_{k,\ell=1}^m.$$

The linear system $(\diamond)_1$ for the Newton increments is explicitly given by

$$
\begin{pmatrix}
G_1 & -I & 0 & \cdots & 0 \\
0 & G_2 & -I & \cdots & \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & G_{m-1} & -I \\
A & 0 & \cdots & 0 & B
\end{pmatrix}
\begin{pmatrix}
\Delta s_1^{(\nu)} \\
\Delta s_2^{(\nu)} \\
\vdots \\
\Delta s_{m-1}^{(\nu)} \\
\Delta s_m^{(\nu)}
\end{pmatrix}
= -
\begin{pmatrix}
F(s^{(1)}) \\
F(s^{(2)}) \\
\vdots \\
F(s^{(m-1)}) \\
F(s^{(m)})
\end{pmatrix},
$$

where $G_k := W(x_{k+1}, x_k)|_{y(x,s_k)}$, $1 \leq k \leq m - 1$, are the Wronski matrices and $A := \frac{\partial r}{\partial y_a}$, $B := \frac{\partial r}{\partial y_b}$. 
Remarks:
(i) Computation of $A, B$ by numerical differentiation,
(ii) Computation of $G_k$ by solving the initial value problem

$$G_k'(x) = f_y(x, y(x)) G_k(x), \quad x \in [x_k, x_{k+1}], \quad 1 \leq k \leq m - 1,$$

$$G_k(x_k) = I$$

using internal numerical differentiation $f_y \mapsto \Delta f / \Delta y$.

Solution of the linear systems for the Newton increments: Two alternatives
(i) Global solution: Codes
RWPM [Hermann/Brandt (1982)]:
Gauss elimination with partial pivoting and post-processing
BOUNDSCO [Oberle (1987)]:
Right Householder transformation; no post-processing
BVPSOG [Deuflhard/Kunkel (1985)]:
Sparse elimination by MA28 [Duff/Harwell]
(ii) Block Gauss elimination with post-processing

Example: $m = 3$

\[(1) \quad G_1 \Delta s_1 - \Delta s_2 = - F_1 / G_2 \]
\[(2) \quad G_2 \Delta s_2 - \Delta s_3 = - F_2 \]
\[(3) \quad A \Delta s_1 + B \Delta s_3 = - r \]

\[(1)' \quad G_2 G_1 \Delta s_1 - G_2 \Delta s_2 = - G_2 F_1 / + \]
\[(2) \quad G_2 \Delta s_2 - \Delta s_3 = - F_2 \]

\[(2)' \quad G_2 G_1 \Delta s_1 - \Delta s_3 = - (F_2 + G_2 F_1) / B \]

\[(2)'' \quad B G_2 G_1 \Delta s_1 - B \Delta s_3 = - B (F_2 + G_2 F_1) / + \]
\[(3) \quad A \Delta s_1 + B \Delta s_3 = - r \]

\[(3)' \quad [A + B G_2 G_1] \Delta s_1 = - r - B (F_2 + G_2 F_1) =: - u \]
General case: We obtain

\((\bullet)\) \quad E \Delta s_1 = -u

where

\[ E := A + B G_{m-1} \ldots G_1, \]
\[ u := r + B [F_{m-1} + G_{m-1} F_{m-2} + \ldots + G_{m-1} \ldots G_2 F_1]. \]

1. Step: Solution of \((\bullet)\) by QR decomposition with rank decision

2. Step: Recursive computation of \(\Delta s_k\), \(2 \leq k \leq m\):

\[ G_k \Delta s_k - \Delta s_{k+1} = -F_k, \quad 1 \leq k \leq m-1 \quad \Rightarrow \]
\[ \Delta s_{k+1} = G_k \Delta s_k + F_k. \]
Post-processing with 'iterative refinement sweeps’

Difficulties in case of standard post-processing

Given iterates $\Delta \tilde{s}^{(\nu)}$, $\nu \geq 0$, and using the same mantissa lengths, compute the residuals

$$
\begin{align*}
\text{dr}^{(\nu)} & := \text{fl}[r + A \Delta \tilde{s}^{(\nu)} + B \Delta \tilde{s}^{(m) \nu}], \\
\text{dF}_k^{(\nu)} & := \text{fl}[G_k \Delta \tilde{s}^{(\nu)} + F_k - \Delta \tilde{s}^{(\nu)}], \quad 1 \leq k \leq m - 1
\end{align*}
$$

and solve the linear system for the corrections as before:

$$
\begin{align*}
\text{E} \text{ds}^{(\nu)}_1 &= - \text{du}^{(\nu)}, \\
\text{du}^{(\nu)} & := \text{dr}^{(\nu)} + B \left[ \text{dF}^{(\nu)}_{m-1} + G_{m-1} \text{dF}^{(\nu)}_{m-2} + \ldots \right], \\
\text{ds}^{(\nu)}_{k+1} &= G_k \text{ds}^{(\nu)}_k + \text{dF}^{(\nu)}_k, \quad 1 \leq k \leq m - 1, \\
\Delta \tilde{s}^{(\nu+1)}_k &= \Delta \tilde{s}^{(\nu)}_k + \text{d}\tilde{s}^{(\nu)}_{k+1}, \quad 1 \leq k \leq m, \\
\text{d}\tilde{s}^{(\nu)}_k & := \text{fl}(\text{ds}^{(\nu)}_k), \quad 1 \leq k \leq m.
\end{align*}
$$

Convergence, if $\varepsilon \sigma(a,b) (m - 1) (2n + m - 1) \ll 1$ with $\sigma(a,b) := \max_{x \in \mathbb{R}} \|W(a,x)\|$. Difficulties to be expected, if $\sigma(a,b) \gg 1$. 
β) Iterative refinement sweeps

Let $\varepsilon$ be the prescribed relative accuracy and

$$\bar{\varepsilon} := \| d\tilde{s}^{(0)}_1 \|$$

[$d\tilde{s}^{(0)}_1$ : result after one post-processing step].

β) If $\bar{\varepsilon} > \varepsilon$, set $k_0 := 0$ and 'switch' to the global solution of the linear block systems.

β) If $\bar{\varepsilon} \leq \varepsilon$, define the sweep index

$$k_{\nu} := \max \{ 1 \leq k \leq m | \| d\tilde{s}^{(\nu)}_j \| \leq \bar{\varepsilon}, 1 \leq j \leq k \}$$

and set

$$dF^{(\nu)}_k = 0, \quad 1 \leq k \leq k_{\nu} - 1.$$ 

Convergence for at most $m - 1$ post-processing steps under the assumption

$$\varepsilon \sigma_{\Delta} (m - 1) (2n + m - 1) \ll 1, \quad \sigma_{\Delta} := \max_{x_k \in \Delta} \sigma(x_k, x_{k+1}).$$

Codes:

BVPSOL [Deuflhard/Bader (1983)] with switching to BVPSOG, if $k_0 = 0$. 