6 Galerkin method, finite differences and collocation

6.1 Galerkin method
Consider a scalar 2nd order ordinary differential equation in selfadjoint form with homogeneous boundary conditions:

\((*)_1\) \quad - (r(x) y'(x))' + s(x) y(x) = f(x), \quad x \in [a, b],

\((*)_2\) \quad y(a) = 0, \quad y(b) = 0,

where \(r, s \in C(I), \ I := [a, b]\) and

\[0 < c_0 \leq r(x) \leq c_1, \quad x \in I, \quad s(x) \geq 0, \quad x \in I.\]

Scalar multiplication of \((*)_1\) by a function \(\varphi \in C^1_0(I)\) and integration over \(I\) yields

\[- \int_a^b (r(x) y'(x))' \varphi(x) \, dx + \int_a^b s(x) y(x) \varphi(x) \, dx = \int_a^b f(x) \varphi(x) \, dx.\]

Partial integration of the first integral results in

\[- \int_a^b (r(x) y'(x))' \varphi(x) \, dx = \int_a^b r(x) y'(x) \varphi'(x) \, dx - (r(x) y'(x) \varphi(x))|_a^b = 0.\]
We thus obtain
\[ \int_{a}^{b} r(x) y'(x) \varphi'(x) \, dx + \int_{a}^{b} s(x) y(x) \varphi(x) \, dx = \int_{a}^{b} f(x) \varphi(x) \, dx. \]

In view of the preceding equation, we introduce a norm on \( C^1_0(I) \) according to
\[ \| y \|_1 := \left( \int_{a}^{b} (y'(x)^2 + y(x)^2) \, dx \right)^{1/2}. \]

The space \((C^1_0(I), \| \cdot \|_1)\) is not complete. We denote by \( H^1_0(I) \) the completion of \((C^1_0(I), \| \cdot \|_1)\) with respect to the topology defined by \( \| \cdot \|_1 \) on \( C^1_1(I) \).

Then, the variational equation
\[ (\diamond) \quad \int_{a}^{b} r(x) y'(x) \varphi'(x) \, dx + \int_{a}^{b} s(x) y(x) \varphi(x) \, dx = \int_{a}^{b} f(x) \varphi(x) \, dx, \quad \varphi \in H^1_0(I) \]
has a unique solution \( y \in H^1_0(I) \).

We choose a subspace \( S_h \subset H^1_0(I) \) with \( \dim S_h = n_h < \infty \) and compute an approximation \( y_h \in S_h \) as the solution of the finite dimensional variational equation
\[ (\diamond)_h \quad \int_{a}^{b} r(x) y_h'(x) \varphi_h'(x) \, dx + \int_{a}^{b} s(x) y_h(x) \varphi_h(x) \, dx = \int_{a}^{b} f(x) \varphi_h(x) \, dx, \quad \varphi_h \in S_h. \]
We choose $S_h \subset H^1_0(I)$ as the linear subspace $S_{1,0}(I; I_h)$ of linear splines with respect to a partition $I_h$ of the interval $I := [a, b]$:

\[ I_h := \{ x_k := a + k h \mid 0 \leq k \leq n + 1 \}, \quad h := \frac{b - a}{n + 1}, \]

\[ S_{1,0}(I; I_h) := \{ v_h \in C_0(I) \mid v_h|_{[x_k, x_{k+1}]} \in P_1([x_k, x_{k+1}]), \ 0 \leq k \leq n \}. \]

The space $S_{1,0}(I; I_h)$ is spanned by the basis functions $\varphi^{(k)}_h$, $1 \leq k \leq n$, as given by

\[ \varphi^{(k)}_h(x_j) = \delta_{jk}, \ 1 \leq j, k \leq n, \]

i.e., $S_{1,0}(I; I_h) = \text{span} \{ \varphi^{(1)}_h, \ldots, \varphi^{(n)}_h \}$.

The solution $y_h \in S_{1,0}(I; I_h)$ can be written as a linear combination of the basis functions

\[ y_h = \sum_{j=1}^{n} c_j \varphi^{(j)}_h. \]

Then, $(\diamond)_h$ holds true if and only if

\[ \sum_{j=1}^{n} \left[ \left\{ \int_{a}^{b} r(x) (\varphi^{(j)}_h)'(x) (\varphi^{(i)}_h)'(x) \, dx \right\} + \left\{ \int_{a}^{b} s(x) (\varphi^{(j)}_h)(x) (\varphi^{(i)}_h)(x) \, dx \right\} \right] c_j = \int_{a}^{b} f(x) (\varphi^{(i)}_h)(x) \, dx, \ 1 \leq i \leq n. \]
\( (\diamond)_h \) represents a linear algebraic system \( Ac = b \).

**Definition 6.1 Stiffness matrix and load vector**

The matrix \( A = (a_{ij})_{i,j=1}^n \) and the vector \( b = (b_1, \ldots, b_n)^T \) are referred to as the **stiffness matrix** resp. the **load vector**.

**Lemma 6.2 Properties of the stiffness matrix**

The stiffness matrix \( A \) is a symmetric, positive definite tridiagonal matrix.

**Lemma 6.3 Numerical quadrature**

If we compute the entries of the stiffness matrix and the components of the load vector by the quadrature formula

\[
\int_a^b g(x) \, dx \approx h \left( g(x_k + \frac{h}{2}) \right) =: h \, g_{k+1/2} ,
\]

we obtain

\[
a_{ii} = \frac{r_{k-1/2}}{h} + \frac{r_{k+1/2}}{h} + \frac{h}{4} s_{k-1/2} + \frac{h}{4} s_{k+1/2} , \quad 1 \leq i \leq n ,
\]

\[
a_{i,i+1} = -\frac{r_{k+1/2}}{h} + \frac{h}{4} s_{k+1/2} , \quad 1 \leq i \leq n - 1 , \quad a_{i-1,i} = -\frac{r_{k-1/2}}{h} + \frac{h}{4} s_{k-1/2} , \quad 2 \leq i \leq n ,
\]

\[
b_i = \frac{h}{2} f_{k-1/2} + \frac{h}{2} f_{k-1/2} .
\]
6.2 Finite difference methods

Introducing the variable

\[ z(x) := r(x) y'(x) , \quad x \in I , \]

the 2nd order ordinary differential equation \((*)_1\) is equivalent to a system of two differential equations of 1st order

\[
\begin{align*}
(**)_1 & \quad y'(x) = z(x)/r(x) , \quad x \in I , \\
(**)_2 & \quad z'(x) = s(x) y(x) - f(x) , \quad x \in I .
\end{align*}
\]

We partition the interval \( I := [a, b] \) into the equidistant grid

\[ I_h := \{ x_k = a + k h | 0 \leq k \leq n + 1 \} \]

of step size \( h := (b - a)/(n + 1) \) and approximate the first derivatives \( y'(x) \) and \( z'(x) \) with respect to \( x_{k+1/2} = x_k + h/2 , \quad 0 \leq k \leq n \), by the 1st order central difference quotients

\[
\frac{1}{h} (y(x_{k+1}) - y(x_k)) \approx \frac{z(x_{k+1/2})}{r(x_{k+1/2})} , \quad \frac{1}{h} (z(x_{k+1}) - z(x_k)) \approx s(x_{k+1/2}) y(x_{k+1/2}) - f(x_{k+1/2}) .
\]
Since we want to compute approximations only in the nodes $x_k$, $0 \leq k \leq n+1$, we replace $z(x_{k+1/2})$ and $y(x_{k+1/2})$ by averaging:

$$z(x_{k+1/2}) \approx \frac{1}{2} (z(x_k) + z(x_{k+1})) , \quad y(x_{k+1/2}) \approx \frac{1}{2} (y(x_k) + y(x_{k+1})) .$$

Moreover, for notational convenience we set

$$r_{k+1/2} := r(x_{k+1/2}) , \quad s_{k+1/2} := s(x_{k+1/2}) , \quad f_{k+1/2} := f(x_{k+1/2}) .$$

Then, the finite difference method is as follows:

Compute grid functions $y_h, z_h \in C(I_h)$ as the solution of

\begin{align*}
(*)_1 \quad y_h(x_{k+1}) &= y_h(x_k) + \frac{h}{2} \frac{z_h(x_k) + z_h(x_{k+1})}{r_{k+1/2}} , \quad 0 \leq k \leq n, \\
(*)_2 \quad z_h(x_{k+1}) &= z_h(x_k) + \frac{h}{2} \frac{y_h(x_k) + y_h(x_{k+1})}{s_{k+1/2}} , \quad 0 \leq k \leq n, \\
(*)_3 \quad y_h(a) &= y_h(b) = 0 .
\end{align*}
Solving \((\star)_2\) for \(z_h(x_{k+1})\) gives
\[(\star)_4 \quad z(x_{k+1}) = -z_h(x_k) + \frac{2r_{k+1/2}}{h} (y_h(x_{k+1}) - y_h(x_k)).\]

The corresponding equation for \(z_h(x_k)\) is given by
\[(\star)_5 \quad z(x_k) = -z_h(x_{k-1}) + \frac{2r_{k-1/2}}{h} (y_h(x_k) - y_h(x_{k-1})).\]

Inserting \((\star)_5\) into \((\star)_4\) yields
\[(\star)_6 \quad z_h(x_{k+1}) - z_h(x_{k-1}) = -\frac{2r_{k-1/2}}{h} (y_h(x_k) - y_h(x_{k-1})) + \frac{2r_{k+1/2}}{h} (y_h(x_{k+1}) - y_h(x_k)).\]

On the other hand, adding \((\star)_2\) for \(k - 1\) and \(k\) gives
\[(\star)_{7a} \quad z_h(x_k) = z_h(x_{k-1}) + \frac{h}{2} s_{k-1/2} (y_h(x_{k-1}) + y_h(x_k)) - h f_{k-1/2} / +\]
\[(\star)_{7b} \quad z_h(x_{k+1}) = z_h(x_k) + \frac{h}{2} s_{k+1/2} (y_h(x_k) + y_h(x_{k+1})) - h f_{k+1/2} \implies\]
\[(\star)_8 \quad \frac{h}{2} s_{k-1/2} y_h(x_{k-1}) + \frac{h}{2} (s_{k-1/2} + s_{k+1/2}) y_h(x_k) + \frac{h}{2} s_{k+1/2} y_h(x_{k+1}) = (z_h(x_{k+1}) - z_h(x_{k-1})) = h f_{k-1/2} + h f_{k+1/2}.\]
Inserting $(\star)_6$ into $(\star)_8$, we obtain

$$(\star)_9 \quad \left( - \frac{2 r_{k-1/2}}{h} + \frac{h}{2} s_{k-1/2} \right) y_h(x_{k-1}) + \left( \frac{2 r_{k-1/2}}{h} + \frac{2 r_{k+1/2}}{h} + \frac{h}{2} s_{k-1/2} + \frac{h}{2} s_{k+1/2} \right) y_h(x_k) + \left( - \frac{2 r_{k+1/2}}{h} + \frac{h}{2} s_{k+1/2} \right) y_h(x_{k+1}) = h \left( f_{k-1/2} + f_{k+1/2} \right).$$

A comparison of $(\star)_9$ and Lemma 6.3 shows:

The Galerkin method with numerical quadrature is equivalent to the finite difference method developed in this subsection.
6.3 Collocation

We consider the boundary value problem

\[(*)_1 \quad -(r(x) y'(x))' + s(x) y(x) = f(x), \quad x \in [a, b],\]
\[(*)_2 \quad y(a) = y(b) = 0\]

under the same assumptions on \(r, s\) as in Chapter 6.1.

As for the Galerkin method, we are looking for an approximation in a finite dimensional space \(V_n = \text{span} \{\varphi_0, \ldots, \varphi_{n+1}\}\), i.e., we use the ansatz

\[(\bullet) \quad y_n(x) = \sum_{j=0}^{n+1} c_j \varphi_j(x).\]

Idea: Assume that the function given by (\bullet) satisfies the two boundary conditions \((*)_2\) as well as the differential equation in \(n\) prespecified points \(\xi_i \in (a, b), \ 1 \leq i \leq n\), which are called collocation points.

Remark 6.4 We require the continuity of the functions \(\varphi_i, 0 \leq i \leq n + 1\), and its derivatives up to second order in the collocation points.
Collocation methods are classified according to the nature of the ansatz functions:

(i) Polynomial collocation (spectral method)
Use of global polynomials as ansatz functions

(ii) Spline collocation
Use of splines of order $m \geq 3$ with respect to a partition $\Delta_n$ of $[a, b]$.

(i) Polynomial collocation
The ansatz functions $\varphi_i$ are chosen as the orthogonal polynomials of degree $i + 1$, $1 \leq i \leq n$, with respect to the inner product $(y_1, y_2)_{0, 1} := \int_a^b y_1(x)y_2(x)dx$ satisfying $\varphi(a) = \varphi(b) = 0$. The collocation points are chosen as the zeroes $a < \xi_1 < \xi_2 < \ldots < \xi_n < b$ of $\varphi_{n+1}$.

Remark 6.5 A disadvantage of polynomial collocation is that the resulting linear algebraic system has a densely populated coefficient matrix.
(ii) Spline collocation
The ansatz functions are chosen as the splines $s_{3,\Delta_n} \in S_{3,\Delta_n}(I)$ of order 3 with respect to an equidistant partition

$$\Delta_n := \{ x_i = a + i \, h \mid 0 \leq i \leq n \} \, , \, h := \frac{b-a}{n}$$

of $I$ with $\dim S_{3,\Delta_n}(I) = n + 2$.

The basis is chosen using quadratic B-splines:

$$B_2^{(i)}(x) = \begin{cases} 
\frac{1}{2} \left( \frac{3}{2} + \frac{1}{h} (x - x^*) \right)^2, & x \in [x_{i-2}, x_{i-1}] \\
\frac{3}{4} - \frac{1}{h} (x - x^*)^2, & x \in [x_{i-1}, x_i] \\
\frac{3}{2} - \frac{1}{h} (x - x^*)^2, & x \in [x_i, x_{i+1}]
\end{cases}$$

with $x^* := x_{i-1} + \frac{h}{2}$.

**Remark 6.6** Since the ansatz functions are only $C^1$-functions, the collocation points must not coincide with the interior nodes. Therefore, we choose:

$$\xi_0 = a \ , \ \xi_i = \frac{1}{2} (x_{i-1} + x_i) \ , \ 1 \leq i \leq n \ , \ \xi_{n+1} = b.$$
Remark 6.7: The collocation conditions in the boundary nodes $x_0 = a$ and $x_{n+1} = b$ are satisfied automatically, if the B-splines $B_2^{(0)}$ and $B_2^{(n+1)}$ with threefold nodes in $x_0$ and $x_{n+1}$ are eliminated from the basis. This leads to the ansatz

$$y_n(x) = \sum_{j=1}^{n} c_j \varphi_h^{(j)} (x).$$

Lemma 6.8 Properties of the resulting coefficient matrix

The ansatz $(\dagger)$ gives rise to a linear algebraic system $Ac = b$ with a symmetric, positive definite tridiagonal matrix $A \in \mathbb{R}^{n \times n}$.

Proof: Every collocation point $\xi_i$ is situated within the support of at most three B-spline basis functions.