Exercise 4 (Initial value problem with discontinuous right-hand side)

Consider the initial value problem
\begin{align*}
  y'(x) &= -\text{sgn}(y(x)), \quad x \geq -1, \\
  y(-1) &= 1.
\end{align*}

(i) Compute the exact solution of the differential equation. Here, exact solution means a piecewise continuously differentiable function which satisfies the differential equation almost everywhere.

(ii) For \( x_i := -1 + ih, i \in \mathbb{N}_0, \) and \( 0 < h < 1 \) consider the explicit Euler method
\begin{align*}
  y_{i+1} &= y_i - h \text{sgn}(y_i), \quad y_0 = 1.
\end{align*}
Show that for \( n \in \mathbb{N} \) with \( nh \leq 1 < (n+1)h \) and all \( k \in \mathbb{N} \) there holds
\begin{align*}
  y_{n+2k} &= y_n, \quad y_{n+2k+1} = y_{n+1}.
\end{align*}

(iii) Give a reason for the oscillating behavior of the approximations. How can you avoid the oscillations?

6 Points

Exercise 5 (Midpoint rule applied to a model problem)

For the numerical integration of the model problem
\begin{align*}
  y'(x) &= \lambda y(x), \quad x \geq a, \quad y(a) = \alpha
\end{align*}
consider the explicit midpoint rule with an initial Euler step using an equidistant grid \( x_i := a + ih, h > 0: \)
\begin{align*}
  y_{i+1} &= y_{i-1} + 2h\lambda y_i, \quad i \geq 1, \\
  y_1 &= y_0 + h\lambda y_0, \\
  y_0 &= \alpha.
\end{align*}

(i) Determine functions \( g_n(\lambda h), 1 \leq n \leq 2, \) such that the sequences
\begin{align*}
  y_i^{(n)} := g_i(\lambda h)y_0
\end{align*}
satisfy the difference equation associated with the explicit midpoint rule.

(ii) Assume \( \lambda \ll 0 \). Does the approximate solution show the same qualitative behavior as the exact solution?

(iii) Determine \( \mu_k, 1 \leq k \leq 2 \), depending on \( \lambda \) and \( h \), such that

\[
y_i = (\mu_1 g_1^i + \mu_2 g_2^i) \alpha .
\]

Derive a modification of the Euler initial step to improve the quality of the method in case \( \lambda \ll 0 \).

6 Points

**Exercise 6 (Lady Windermere's fan)**

Assume that \( f : I \times D \to D, I := [a, b] \subset \mathbb{R}, D \subset \mathbb{R}^m \), is continuous on \( I \times D \) and satisfies the Lipschitz condition

\[
\| f(x, y_1) - f(x, y_2) \| \leq L \| y_1 - y_2 \|, \ x \in I , \ y_1, y_2 \in D , \ L > 0 .
\]

For the numerical integration of the initial value problem

\[
y'(x) = f(x, y(x)) , \ x \in I := [a, b] \subset \mathbb{R} , \ y(a) = \alpha \in D
\]

consider the explicit one-step method

\[
y_{i+1} = y_i + h_i \Phi(x_i, y_i, h_i; f) ,
\]

\[
y_0 = \alpha ,
\]

where \( a = x_0 < x_1 < ... < x_N = b \) is a not necessarily equidistant partition of \( I \) with step sizes \( h_i := x_{i+1} - x_i, 0 \leq i \leq N - 1 \).

Prove the following assertion: If the one-step method is consistent of order \( p \in \mathbb{N} \), i.e.,

\[
\| \tau h(x, y) \| \leq Ch_{\text{max}}^p , \ x \in I , \ y \in D ,
\]

where \( h_{\text{max}} := \max_{0 \leq i \leq N-1} (x_{i+1} - x_i) \) and \( C \in \mathbb{R}_+ \) is independent of \( h_{\text{max}} \), then there exists a constant \( C' \in \mathbb{R}_+ \), independent of \( h_{\text{max}} \), such that

\[
\| y_N - y(b) \| \leq C' \exp(L(b - a) - 1) \frac{h_{\text{max}}^p}{L}.
\]

6 Points

[Hint: Use the technique known as Lady Windermere’s fan which is illustrated in the figure below:]

![Diagram of Lady Windermere's fan](image-url)
Construct $N$ initial value problems of the form
\[
y'(x) = f(x, y(x)) \ , \ x \geq x_i \ , \ 0 \leq i \leq N - 1 ,
y(x_i) = y_i
\]
with the solutions $y(x; x_i, y_i)$. The quantities $e_i := \|y(x_i; x_{i-1}, y_{i-1}) - y_i\|$ are available by means of the local discretization error. Interpreting $y(x; x_i, y_i)$ as the solution of the perturbed initial value problem
\[
y'(x) = f(x, y(x)) \ , \ x \geq x_i \ , \ 0 \leq i \leq N - 1 ,
y(x_{i+1}) = y_{i+1} + \underbrace{h_i \tau_{h_i}(x_i, y_i)}_{\text{perturbation}} ,
\]
the error propagation can be accessed by Gronwall’s lemma.

Delivery of the homework at latest on February 22, 2010. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.