Numerical Analysis II
(Homework 4)

Exercise 10 (Consistency of linear multi-step methods)

Assume \( f \in C^p(I \times D), I := [a, b] \subset \mathbb{R}, D \subset \mathbb{R}^d \) and \( \alpha \in \mathbb{R}^d \). Consider the initial-value problem

\[
(IVP_1) \quad y'(x) = f(x, y(x)) , \quad x \in I ,
(IVP_2) \quad y(a) = \alpha .
\]

For a given equidistant partition \( x_j := a + jh, h := (b - a)/N, N \in \mathbb{N} \) of \( I \) and given real numbers \( \alpha_k, \beta_k, 0 \leq k \leq m, m \in \mathbb{N} \), with \( \alpha_m \neq 0 \) and vectors \( \alpha^{(j)} \in \mathbb{R}^d, 0 \leq j \leq m - 1 \), a linear multi-step method is of the form

\[
(MSM_1) \quad \frac{1}{h} \sum_{k=0}^{m} \alpha_k y_{j+k} = \sum_{k=0}^{m} \beta_k f(x_{j+k}, y_{j+k}) , \quad 0 \leq j \leq N - m ,
(MSM_2) \quad y_j = \alpha^{(j)} , \quad 0 \leq j \leq m - 1 .
\]

Denoting by \( \tau_h \) the local discretization error, suppose that

\[
(\ast) \quad \tau_h(x_j) = O(h^p) , \quad 0 \leq j \leq m - 1 .
\]

Show that the linear multi-step method \( (MSM)_1, (MSM)_2 \) is consistent with the initial-value problem \((IVP)_1, (IVP)_2\) of order \( p \) if and only if it is consistent of order \( p \) with

\[
(IVP'_1) \quad y'(x) = y(x) , \quad x \in I ,
(IVP'_2) \quad y(a) = 1 .
\]

6 Points

Exercise 11 (Stability of multi-step methods)

In class, the stability of multi-step methods has been investigated by means of the Lipschitz stability of operators on the linear space of grid functions. This approach can be generalized as follows:

Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be normed spaces and let \((A_h)_h\) be a family of continuous operators \( A_h : X \to Y \). This family is called Lipschitz stable, if there exist positive numbers \( \delta, \eta \) such that for all \( x, z \in X \) and all \( h \) satisfying

\[
(LS)_1 \quad \| A_h x - A_h z \|_Y \leq \delta
\]
there holds
\[(LS)_2 \quad \|x - z\|_X \leq \eta \|A_h x - A_h z\|_Y .\]

(i) For Lipschitz stable families of continuous linear operators \(A_h : X \to Y\)
derive a simpler definition of stability by taking advantage of the linearity.

(ii) Let \((A_h)_h\) be a Lipschitz stable family of continuous linear operators which
are additionally assumed to be surjective.
Show that for all \(h\) the operator \(A_h\) is invertible with \(\|A_h^{-1}\| \leq C, C > 0\),
and derive an upper bound for the constant \(C\).
(Hint: Show injectivity first and then use the definition of the operator norm
\(\|A_h\| = \sup_{0 \neq x \in X} \frac{\|A_h x\|_Y}{\|x\|_X}\))

(iii) Let \(H\) be a Hilbert space, i.e., a complete inner product space with inner
product \(< \cdot, \cdot > : H \times H \to \mathbb{R}\), and let \((A_h)_h\) be a family of continuous endomor-
phisms \(A_h : H \to H\). Further, let \(a_h : H \times H \to \mathbb{R}\) be the bilinear form given by
\(a_h(x, y) := \langle A_h x, y \rangle\).
Prove the equivalence of the following two statements:
\((*)\) For all \(h\) the operator \(A_h\) is invertible such that \(\|A_h^{-1}\| \leq C\), where \(C\) is a
positive constant independent of \(h\).
\((***)\) There holds
\[
\sup_{\|y\|=1} |a_h(x, y)| > \frac{1}{C} \|x\| , \quad x \in H ,
\sup_{\|x\|=1} |a_h(x, y)| > 0 , \quad 0 \neq y \in H .
\]
(Hint: For \((*) \Rightarrow (***)\) use \(\|x\| = \sup\{< x, y > \mid \|y\| = 1\}, x \in H\). For
\((***) \Rightarrow (*)\) verify first that \(\|A_h x\| \geq C^{-1} \|x\|, x \in H\). Then, using an arbitrary
Cauchy sequence, show that the range of each operator \(A_h\) is closed. Finally,
prove surjectivity by contradiction, i.e., assume range\((A_h)\) \(\neq \{0\}\).

6 Points

Delivery of the homework at latest on March 29, 2010. The homework
may be submitted either electronically (rohop@math.uh.edu) or as a
hardcopy in class.