Numerical Mathematics II  
(6th Homework Assignment)

Exercise 16 (Cauchy’s \( \theta \)-method)
Consider the initial value problem
\[
(*) \quad y'(x) = f(x, y(x)) , \ x \geq a , \ y(a) = \alpha ,
\]
where \( f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) and \( \alpha \in \mathbb{R}^m \).
Show that the only consistent linear one-step methods for the numerical solution of (*) are given by the one-parameter family
\[
y_h(x_{k+1}) = y_h(x_k) + h \left( (1 - \theta) f(x_k, y_h(x_k)) + \theta f(x_{k+1}, y_h(x_{k+1})) \right) .
\]
(i) For which \( \theta \in \mathbb{R} \) the order of consistency \( p = 2 \) can be achieved?
(ii) Which are the methods obtained for \( \theta = 0, \theta = 0.5 \) and \( \theta = 1.0 \)?
Points: 4

Exercise 17 (Runge Kutta Fehlberg 1(2) methods)
For the numerical integration of the initial value problem
\[
y'(x) = f(x, y(x)) , \ y(a) = \alpha
\]
consider the Runge Kutta steps of order 1 and 2
\[
y_h(a + h) = y_h(a) + h \left( b_1 k_1 + b_2 k_2 \right) , \quad (0.1)
y(a + h) - y_h(a + h) = O(h^2) \quad (0.2)
\]
\[
\hat{y}_h(a + h) = \hat{y}_h(a) + h \left( \hat{b}_1 k_1 + \hat{b}_2 k_2 + \hat{b}_3 k_3 \right) , \quad (0.3)
y(a + h) - \hat{y}_h(a + h) = O(h^3) \quad (0.4)
\]
with step size \( h \) and the increments
\[
\begin{align*}
  k_1 &= f(a, \alpha) , \\
  k_i &= f(a + c_i h, \alpha + h \sum_{j=1}^{i-1} a_{ij} k_j) , \quad 2 \leq i \leq 3 .
\end{align*}
\]

(i) Derive the order conditions for the coefficients \( c_i, a_{ij}, b_i \) and \( \hat{b}_i \). Observe that here \( c_i = \sum_j a_{ij} \) is not required.

(ii) How many function evaluations per step are required by (0.1),(0.2) and (0.3),(0.4)?

(iii) The embedded method that we are going to construct uses Fehlberg’s idea to proceed with \( y_h \) in the next step (Fehlberg’s trick): One takes advantage of the resulting freedom in the choice of the parameters to reuse increment \( k_3 \) as the increment \( k_1 \) in the following step which results in reduced computational work following the first step. Give the conditions on the coefficients \( c_i, a_{31} \) and \( a_{32} \) that guarantee feasibility of Fehlberg’s trick.

(iv) Determine the coefficient of the lowest \( h \) power in the local discretization error of the method (0.3),(0.4) including Fehlberg’s trick from (iii) and the simplification \( \hat{b}_2 = b_2 \).

What are the consequences of the choice \( c_2 = \frac{1}{2}, b_2 = 1 \), taking into account that embedded methods are aimed for a step size control?

(v) The potential benefits of the RKF-1(2) method have not been exhausted by Fehlberg’s trick alone: Determine the parameters such that the result is a Runge Kutta Fehlberg method of order 1 with only one function evaluation per integration step.

(vi) Sketch the method from part (v) in algorithmic notation along with the following step size control: an integration step has to be repeated with the computed smaller step size, if the estimated error \( EST \) exceeds the user-specified tolerance \( TOL \). Otherwise, proceed to the next step with the step size suggested by the step size control.

In order to prevent too large oscillations of the step size, set
\[
h_{\text{new}} := h \min (\text{facmax}, \max (\text{facmin}, \text{fac}(TOL/EST)^{1/(p+1)}))
\]
with a safety factor \( \text{fac} := 0.9 \) as well as the limiters
\[
\text{facmin} := 0.5 , \quad \text{facmax} := 2.0 .
\]

12 Points

Delivery of the homework at latest on March 28, 2006. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.