3 Parabolic Differential Equations

3.1 Classical solutions

We consider existence and uniqueness results for initial-boundary value problems for the linear heat equation in $Q := \Omega \times (0,T)$, where $\Omega$ is a bounded domain in $\mathbb{R}^d$ with boundary $\Gamma := \partial \Omega$ and $T \in \mathbb{R}_+$. For given functions

\[(3.1.1) \quad f \in C(Q), \quad u^D \in C(\Gamma \times (0,T)), \quad u^0 \in C(\Omega),\]

satisfying the compatibility condition

\[(3.1.2) \quad u^D(x,0) = u^0(x), \quad x \in \Gamma,\]

we look for a function $u : Q \to \mathbb{R}$ such that

\[(3.1.3a) \quad u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in Q,\]
\[(3.1.3b) \quad u(x,t) = u^D(x,t), \quad (x,t) \in \Gamma \times (0,T),\]
\[(3.1.3c) \quad u(x,0) = u^0(x), \quad x \in \Omega.\]

\[(3.1.4) \quad \text{Definition: Under the assumptions (3.1.1),(3.1.2) a function } u \in C(Q) \text{ such that } u(x,\cdot) \in C^1((0,T)) \text{ for } x \in \Omega, \text{ and } u(\cdot,t) \in C^2(\Omega) \text{ for } t \in (0,T), \text{ is called a classical solution of the initial-boundary value problem (3.1.3a)-(3.1.3c)} \text{, if } u \text{ satisfies (3.1.3a)-(3.1.3c) pointwise.} \]

The initial-boundary value problem (3.1.3a)-(3.1.3c) is said to be well-posed, if its solution $u$ depends continuously on the data $f, u^D, u^0$ with respect to the topology defined by some norm $\| \cdot \|$ on $C(Q)$.

In the following, we summarize the most important existence and uniqueness results for classical solutions of the initial-boundary value problem (3.1.3a)-(3.1.3c). For proofs and further details we refer to the literature (cf., e.g., Jost (2002)).

The uniqueness of a classical solution follows from the weak maximum principle.

\[(3.1.5) \quad \text{Theorem: Assume that (3.1.1) and (3.1.2) hold true and denote by } \partial_R Q \text{ the reduced boundary } \partial_R Q := (\overline{\Omega} \times \{0\}) \cup (\Gamma \times [0,T]). \text{ Moreover, suppose} \]
that the function $u \in C(Q)$ such that $u(x, \cdot) \in C^1((0, T))$ for $x \in \Omega$ and $u(\cdot, t) \in C^2(\Omega)$ for $t \in (0, T)$ satisfies the inequality

$$\frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) \leq 0 , \quad (x, t) \in Q .$$

Then, there holds

$$\max_{(x,t)\in Q} u(x,t) = \max_{(x,t)\in \partial RQ} u(x,t) .$$

If $u_1, u_2$ are two classical solutions, the application of Theorem (3.1.5) to $u_1 - u_2$ and $u_2 - u_1$ implies $u_1 = u_2$, i.e., we have:

**Corollary** (10.4.1.6) Under the assumptions (3.1.1), (3.1.2) the initial-boundary value problem (3.1.3a)-(3.1.3c) admits a unique classical solution.

**Remark** (10.4.1.7) In case $\Omega = \mathbb{R}^d$, the boundary condition (3.1.3b) has to be replaced by the growth condition

$$u(x, t) = O(\exp(\lambda |x|^2)) \quad \text{for } |x| \to \infty \quad (\lambda > 0) .$$

Besides the weak maximum principle there holds the following *strong maximum principle*:

**Theorem:** Under the assumptions of Theorem (3.1.5) suppose that the function $u \in C(Q)$ such that $u(x, \cdot) \in C^1((0, T))$ for $x \in \Omega$ and $u(\cdot, t) \in C^2(\Omega)$ for $t \in (0, T)$ satisfies the inequality

$$\frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) \leq 0 , \quad (x, t) \in Q .$$

If there exists a point $(x_0, t_0) \in Q$ such that

$$u(x_0, t_0) = \max_{(x,t)\in Q} u(x,t) ,$$

there holds

$$u(x, t) = \text{const.} , \quad (x, t) \in \overline{\Omega} \times [0, t_0] .$$

The existence of a classical solution of the initial-boundary value problem (3.1.3a)-(3.1.3c) follows from an explicit representation with respect to the data $f, u^D, u^0$ which can be given by means of a kernel function of the linear heat equation.

**Definition:** Let $\Omega$ be a domain in $\mathbb{R}^d$. A function $K : \overline{\Omega} \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is called a kernel function of the linear heat equation, if there holds

\begin{enumerate}[(i)]
    \item \[ \frac{\partial}{\partial t} K(x, y, t) - \Delta_x K(x, y, t) = 0 , \quad (x, y, t) \in \Omega \times \Omega \times \mathbb{R}_+ , \]
    \item \[ K(x, y, t) = 0 , \quad (x, y, t) \in \Gamma \times \Omega \times \mathbb{R}_+ , \]
    \item \[ \lim_{t \to 0} \int_{\Omega} K(x, y, t)v(x) \, dx = v(y) , \quad y \in \Omega , \quad v \in C(\Omega) . \]
\end{enumerate}
The existence of a kernel function can be shown by means of the fundamental solution
\[ \Lambda(x, y, t) := (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \]
of the linear heat equation:
Suppose that \( u^0 \in C(\mathbb{R}^d) \) is bounded and let \( u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \) be the function defined by the convolution
\[ u(x, t) = \Lambda(x, \cdot, t) * u^0(\cdot) = \int_{\mathbb{R}^d} \Lambda(x, y, t)u^0(y) \, dy. \]
Then \( u \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+) \), and \( u \) is a classical solution of the linear heat equation in \( (x, t) \in \mathbb{R}^d \times \mathbb{R}_+ \). Indeed, computing the partial derivatives \( \partial \Lambda/\partial t \) and \( \partial^2 \Lambda/\partial x_i \partial x_j, 1 \leq i, j \leq d \), it is easy to show that \( \Lambda \), considered as a function of \( x \), satisfies the linear heat equation in \( (x, t) \in \mathbb{R}^d \times \mathbb{R}_+ \). It follows that
\[ \frac{\partial}{\partial t} u(x, t) = \left( \frac{\partial}{\partial t} \Lambda(x, \cdot, t) \right) * u^0(\cdot) = (\Delta_x \Lambda(x, \cdot, t)) * u^0(\cdot) = \Delta_x u(x, t). \]
Since \( \Lambda \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+) \), differentiation implies that \( u \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+) \).

For a bounded \( C^2 \) domain \( \Omega \subset \mathbb{R}^d \), one can also use the fundamental solution to obtain an explicit representation of the solution of the initial-boundary value problem (3.1.3a)-(3.1.3c) under a homogeneous initial condition, i.e., \( u^0 \equiv 0 \), and thus deduce the existence of a kernel function:

(3.1.8) Theorem: Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain. Then, the linear heat equation has a kernel function \( K \in C^\infty(\Omega \times \Omega \times \mathbb{R}_+) \) such that \( K(x, y, t) > 0 \) and \( K(x, y, t) = K(y, x, t), (x, y, t) \in \Omega \times \Omega \times \mathbb{R}_+ \).

Using the kernel function \( K \) from Theorem (3.1.8) we obtain the following representation of the unique solution of the initial-boundary value problem (3.1.3a)-(3.1.3c):

(3.1.9) Theorem: Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain and suppose that the assumptions (3.1.1), (3.1.2) hold true. Then, the initial-boundary value problem (3.1.3a)-(3.1.3c) admits a unique solution \( u \in C(\overline{Q}) \cap C^\infty(Q) \) which has the representation
\[ u(x, t) = \int_0^t \int_{\Omega} K(x, y, t-s)f(y, s) \, dy \, ds + \int_{\Omega} K(x, y, t)u^0(y) \, dy + \int_{\Gamma} \nu_\Gamma \cdot \nabla_y K(x, y, t-s)u^D(y, s) \, d\sigma(y) \, ds. \]
Remark (10.4.1.12) If the compatibility condition (3.1.2) does not hold true, one can establish the existence of a unique solution \( u \in C((\overline{\Omega} \times \mathbb{R}_+) \setminus (\Gamma \times \{0\})) \cap C^\infty(Q) \) with the same representation as in Theorem (3.1.9).

Theorem (3.1.9) shows the well-posedness of the initial-boundary value problem (3.1.3a)-(3.1.3c). It also reflects the smoothing property of parabolic differential operators. The solution of the initial-boundary value problem is \( C^\infty \) in \( Q \) even if the initial-boundary data are only continuous.

### 3.2 Finite difference methods

We consider the following initial-boundary value problem for a parabolic differential equation

\[
(3.2.1a) \quad Lu := u_t + A(t)u = f \quad \text{in} \quad Q := \Omega \times [0,T],
\]

\[
(3.2.1b) \quad u = u^D \quad \text{on} \quad \Gamma \times (0,T),
\]

\[
(3.2.1c) \quad u(\cdot,0) = u^0 \quad \text{in} \quad \Omega.
\]

Here, \( \Omega \subset \mathbb{R}^d \) stands for a bounded Lipschitz domain with boundary \( \Gamma = \partial \Omega \) and \( T > 0 \). Further, for any \( t \in (0,T) \) we suppose that \( A(t) \) is a linear second order elliptic differential operator. Moreover, we assume \( f \in C(Q), u^D \in C(\Gamma \times [0,T]) \) and \( u^0 \in C(\overline{\Omega}) \) and that (3.2.1a)-(3.2.1c) is well-posed.

The numerical solution can be based on a semi-discretization either in space or in time. The first approach is referred to as the method of lines. We assume a grid point set \( (3.2.2) \)

\[
\Omega_h := \{ x = (x_{\nu_1}, \cdots, x_{\nu_d})^T \}.
\]

and define \( \Omega_h \) and \( \Gamma_h \) as the sets of interior and boundary grid points according to \( \Omega_h := \overline{\Omega}_h \cap \Omega \) and \( \Gamma_h := \overline{\Omega}_h \cap \Gamma \). Moreover, we assume \( A_h(t), t \in (0,T), \) to be a compact finite difference approximation of \( A(t) \) as in chapter 2.2 and suppose that \( f_h(t) \in C(\overline{\Omega}_h), t \in (0,T], u^D_h \in C(\Gamma_h) \) and \( u^0_h \in C(\overline{\Omega}_h) \). Then, the method of lines amounts to the solution of the initial-value problem

\[
(3.2.3a) \quad (u_h)_t + A_h(t)u_h = f_h(t) \quad , \quad t \in (0,T),
\]

\[
(3.2.3b) \quad u_h(0) = u^0_h
\]

for a system of first-order ordinary differential equations in the unknowns \( u_h(x), x \in \Omega_h \). This system is typically stiff so that the numerical integration of (3.2.3a),(3.2.3b) has to be based on implicit or semi-implicit schemes. The second approach is called Rothe’s method. Discretizing (3.2.1a) implicitly in time, e.g. by the backward Euler scheme, with respect to a uniform partition

\[
(3.2.4) \quad \mathcal{T}_k := \{ t_m := mk , \quad k := T/(M+1) , \quad M \in \mathbb{N} \}
\]
of the time interval \([0, T]\), at each time step we have to solve an elliptic boundary value problem of the form

\[
(3.2.5a) \quad (I + kA(t_k))u(t_k) = u(t_k-1) + kf(t_k) \quad \text{in} \; \Omega,
\]

\[
(3.2.5b) \quad u(\cdot, t_k) = u^D(t_k) \quad \text{on} \; \Gamma.
\]

Consequently, the techniques of chapter 2.2 and 2.4 can be applied for the numerical solution of (3.2.5a), (3.2.5b).

In the sequel, we will focus on finite difference methods in time and space. We set \(Q_{h,k} := Q_h \times I_k\). For \(D_{h,k} \subseteq Q_{h,k}\) we refer to \(C(D_{h,k})\) as the linear space of grid functions on \(D_{h,k}\).

For grid functions \(u_{h,k} \in C(Q_{h,k})\) we define the forward difference quotient

\[
(3.2.6) \quad D^+_h u_{h,k}(x, t) := k^{-1}\left(u_{h,k}(x, t + k) - u_{h,k}(x, t)\right),
\]

where \(x \in \Omega_h\), \(t \in I_k \setminus \{T\}\), and the backward difference quotient

\[
(3.2.7) \quad D^- h u_{h,k}(x, t) := k^{-1}\left(u_{h,k}(x, t) - u_{h,k}(x, t - k)\right),
\]

where \(x \in \Omega_h\), \(t \in I_k \setminus \{0\}\). Moreover, let \(A_h(t), t \in I_k\), be a difference approximation of \(A(t)\) as in chapter 2.3 and suppose that \(f_{h,k} \in C(Q_{h,k}), u^D_h \in C(I_h \times I_k)\) and \(u^0_h \in C(\overline{Q}_h)\).

Using the forward difference quotient, we obtain the difference operator \(L_{h,k} := D^+_h + A_h(t)\) and thus the explicit finite difference method

\[
(3.2.8a) \quad L_{h,k} u_{h,k} = f_{h,k} \quad \text{in} \; \Omega_h \times (I_k \setminus \{T\}),
\]

\[
(3.2.8b) \quad u_{h,k} = u^D \quad \text{auf} \; \Gamma_h \times I_k,
\]

\[
(3.2.8c) \quad u_{h,k} = u^0 \quad \text{in} \; \Omega_h.
\]

The use of the backward difference quotient leads to the difference operator \(L_{h,k} := D^- h + A_h(t)\) and the implicit finite difference method

\[
(3.2.9a) \quad L_{h,k} u_{h,k} = f_{h,k} \quad \text{in} \; \Omega_h \times (I_k \setminus \{0\}),
\]

\[
(3.2.9b) \quad u_{h,k} = u^D \quad \text{auf} \; \Gamma_h \times I_k,
\]

\[
(3.2.9c) \quad u_{h,k} = u^0 \quad \text{in} \; \Omega_h.
\]

If we know \(u_{h,k}(x, t)\) in \(x \in \Omega_h, t = t_m, 0 \leq m \leq M\), in case of the explicit finite difference method we obtain the approximation \(u_{h,k}(x, t)\) in \(x \in \Omega_h, t = t_{m+1}\), simply by solving (3.2.9) for \(u_{h,k}(x, t_{m+1})\). On the other hand, for the implicit finite difference method the computation of \(u_{h,k}(x, t_m), x \in \overline{Q}_h\), requires the solution of a linear algebraic system.

The multiplication of (3.2.9a) by some \(\Theta \in [0, 1]\) (for \(t+k\) instead of \(t\)) and of (3.2.8a) by \(1 - \Theta\) and the subsequent addition of the resulting difference equations results in the so-called \(\Theta\)-method:
where \( L_{h,k}^{\Theta} \) stands for the difference operator
\[
L_{h,k}^{\Theta} u_{h,k} := k^{-1}(u_{h,k}(\cdot, t + k) - u_{h,k}(\cdot, t)) + \\
\quad + \Theta A_h(t + k)u_{h,k}(\cdot, t + k) + (1 - \Theta)A_h(t)u_{h,k}(\cdot, t)
\]
and \( f_{h,k}^{\Theta} \) for the grid function
\[
f_{h,k}^{\Theta} := \Theta f_{h,k}(\cdot, t + k) + (1 - \Theta)f_{h,k}(\cdot, t).
\]
For \( \Theta = 0 \), we obtain the explicit finite difference method (3.2.8a)-(3.2.8c), whereas \( \Theta = 1 \) results in the implicit finite difference method (3.2.9a)-(3.2.9c). In case \( \Theta = 1/2 \), we refer to (3.2.10a)-(3.2.10c) as the Crank-Nicolson method.

We study the convergence of the \( \Theta \)-scheme within the framework of a more general finite difference approximation of (3.2.1a)-(3.2.1c) using a two-level (one-step) finite difference method, i.e., a method involving only two subsequent time levels \( t_{j-1} \) and \( t_j \). To this end let \( M_h(t_j), N_h(t_{j-1} : C(\Omega_h) \to C(\overline{\Omega}_h) \) be two finite difference approximations of \( A(t_j) \) and \( A(t_{j-1}) \), respectively, and assume \( g_h(\cdot, t_{j-1}, t_j) \in C(\Omega_h) \). Moreover, let \( L_{h,k} : C(\overline{\Omega}_{h,k}) \to C(\overline{\Omega}_{h,k}) \) be the finite difference operator given by
\[
L_{h,k} u_{h,k}(\cdot, t_j) := (I_{h,k} + kM_h(t_j))u_{h,k}(\cdot, t_j) - \\
- (I_{h,k}-)kN_h(t_{j-1})u_{h,k}(\cdot, t_{j-1})
\]
We consider the two-level scheme
\[
L_{h,k} u_{h,k}(x, t) = g_{h,k}(x, t), \quad x \in \Omega_h, \quad t \in \overline{I}_k \setminus \{0\},
\]
\[
u_{h,k}(x, t) = u_h^{D}(x, t), \quad x \in I_h, \quad t \in \overline{I}_k \setminus \{0\},
\]
\[
u_{h,k}(x, 0) = u^{0}_h(x), \quad x \in \overline{\Omega}_h.
\]

For the specification of the convergence we measure the global discretization error \( u|_{\overline{\Omega}_{h,k}} - u_{h,k} \) in a suitable discrete norm \( || \cdot ||_{h,k} \) of the linear space \( C(\overline{\Omega}_{h,k}) \) of grid functions on \( \overline{\Omega}_{h,k} \). For instance, we may choose the discrete maximum norm
\[
||u_{h,k}||_{h,k} := \max_{(x,t) \in \overline{\Omega}_{h,k}} |u_{h,k}(x, t)|
\]
or the discrete analogue of the \( L^2([0, T]; L^2(\Omega)) \)-norm (cf. chapter 3.3)
\[
||u_{h,k}||_{h,k} := \left( k \sum_{j=0}^{M} ||u_{h,k}(\cdot, t_j)||_{2,h}^2 \right)^{1/2},
\]
where \( || \cdot ||_{2,h} \) stands for the discrete \( L^2 \)-norm on \( C(\overline{\Omega}_h) \).
\( (3.2.13) \) Definition: Assume that \( u \) is the classical solution of the initial-boundary value problem (3.2.1a)-(3.2.1c) and that \( u_{h,k} \in C(\Omega_{h,k}) \) is the solution of the finite difference method (3.2.12a)-(3.2.12c). The finite difference method is called convergent, if there holds

\[
\| u - u_{h,k} \|_{h,k} \to 0 \quad \text{for} \quad h, k \to 0 .
\]

It is said to be convergent of order \( p_1 \) in \( h \) and \( p_2 \) in \( k \) \( (p_i \in \mathbb{N}, 1 \leq i \leq 2) \), if there exists a constant \( C \geq 1 \), independent of \( h \) and \( k \), such that

\[
\| u - u_{h,k} \|_{h,k} \leq C (h^{p_1} + k^{p_2}) .
\]

As in the numerical solution of initial-value problems for ordinary differential equations, sufficient conditions for convergence are given by the consistency and the stability.

\( (3.2.14) \) Definition: Under the assumptions of Definition (3.2.13), the grid function

\[
\tau_{h,k}(x,t) := \begin{cases} 
L_{h,k} u(x,t) - g_{h,k}(x,t) , & x \in \Omega_h , \ t \in I_k^* \\
u(x,t) - u_{h,k}^D(x,t) , & x \in \Gamma_h , \ t \in I_k \\
u(x,0) - u_0^D(x) , & x \in \Omega_h 
\end{cases}
\]

is called the local discretization error. The finite difference method (3.2.12a)-(3.2.12c) is called consistent with the initial-boundary value problem (3.2.1a)-(3.2.1c), if there holds

\[
\| \tau_{h,k} \|_{h,k} \to 0 \quad \text{for} \quad h, k \to 0 .
\]

The finite difference method is said to be consistent of order \( p_1 \) in \( h \) and \( p_2 \) in \( k \) \( (p_i \in \mathbb{N}, 1 \leq i \leq 2) \), if there exists a positive constant \( C \), independent of \( h \) and \( k \), such that

\[
\| \tau_{h,k} \|_{h,k} \leq C (h^{p_1} + k^{p_2}) .
\]

Assuming a sufficiently smooth classical solution \( u \) of (3.2.1a)-(3.2.1c), consistency and order of consistency can be shown by Taylor expansion.

Example. We consider the \( \Theta \)-method (3.2.10a)-(3.2.10c) applied to the linear heat equation, i.e., \( A(t) = -\Delta, t \in [0,T] \), and \( A_h(t) = -\Delta_h, t \in [0,T] \), where \( \Delta_h \) stands for the discrete Laplace operator, and \( u_h^D = u^D|_{\Gamma_h \times I_k}, u_h^0 = u^0|_{\Omega_h} \).

We assume that the classical solution satisfies \( u \in C^2([0,T];C^4(\overline{T})) \). Then, Taylor expansion reveals that for \( \Theta \neq 1/2 \) the \( \Theta \)-method is consistent of order \( p_1 = 2 \) in \( h \) and \( p_2 = 1 \) in \( k \). We note that the expansion point is \( (x,t) \) for the explicit method and \( (x,t+k) \) for the implicit methods.

If the classical solution \( u \) satisfies \( u \in C^3([0,T];C^4(\overline{T})) \), the Crank-Nicolson method \( (\Theta = 1/2) \) is consistent of order \( p_1 = 2 \) in \( h \) and \( p_2 = 2 \) in \( k \). As the expansion point one has to choose \( (x,t+k/2) \).

The stability of the difference approximation (3.2.12a)-(3.2.12c) means continuous dependence of the solution \( u_{h,k} \) on the data \( g_{h,k}, u_{h,k}^D, u_{h,k}^0 \).
Definition: Let $u_{h,k}$ and $z_{h,k}$ be the solutions of the finite difference method (3.2.12a)-(3.2.12c) with respect to the data $g_{h,k}, u_{h,k}^D, u_h^0$ and the perturbed data $g_{h,k} + \delta g_{h,k}, u_{h,k}^D + \delta u_{h,k}^D, u_h^0 + \delta u_h^0$ with grid functions $\delta g_{h,k} \in C(\Omega_h), \delta u_{h,k}^D \in C(\Gamma_h \times \bar{I}_k), \delta u_h^0 \in C(\bar{\Omega}_h)$. Then, the finite difference method (3.2.12a)-(3.2.12c) is said to be stable, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, h, k) > 0$ such that

$$\|u_{h,k} - z_{h,k}\|_{h,k} < \varepsilon$$

for all perturbations $\delta u_h^0$ and $(\delta g_{h,k}, \delta u_{h,k}^D)$ satisfying

$$\|\delta u_h^0\|_h + \|(\delta g_{h,k}, \delta u_{h,k}^D)\|_{h,k} < \delta.$$

As in the case of finite difference methods for elliptic boundary value problems (see chapter ??), the stability of consistent finite difference approximations is a sufficient condition for convergence.

Theorem: Assume that (3.2.12a)-(3.2.12c) is a finite difference method being consistent with the well-posed initial-boundary value problem (3.2.1a)-(3.2.1c). If it is stable, then it is convergent, and the order of convergence is the same as the order of consistency.

Proof. The solution $u$ of the initial-boundary value problem (3.2.1a)-(3.2.1c) satisfies the equations

$$L_{h,k}u(x,t) = g_{h,k}(x,t) + \tau_{h,k}(x,t), \quad x \in \Omega_h, \quad t \in \bar{I}_k \setminus \{0\},$$

$$u(x,t) = u_{h,k}^D(x,t) + \tau_{h,k}(x,t), \quad x \in \Gamma_h, \quad t \in \bar{I}_k \setminus \{0\},$$

$$u(x,0) = u_h^0(x) + \tau_{h,k}(x,0), \quad x \in \bar{\Omega}_h.$$

Setting $\delta g_{h,k}(x,t) = \tau_{h,k}(x,t), x \in \Omega_h, t \in \bar{I}_k \setminus \{0\}, \delta u_{h,k}^D = \tau_{h,k}(x,t), x \in \Gamma_h, t \in \bar{I}_k \setminus \{0\},$ and $\delta u_h^0(x) = \tau_{h,k}(x,0), x \in \bar{\Omega}_h$, and observing $\|\tau_{h,k}\|_{h,k} \to 0$ for $h, k \to 0$, the assertion follows from the stability (3.2.15).

Remark (9.3.2.17) In the literature, Theorem (3.2.16) is sometimes referred to as the Theorem of Lax (cf., e.g., Thomas (1995)). Under the same assumptions, it can be further shown that the stability also constitutes a necessary condition for the convergence of a consistent finite difference method. The equivalence of the stability and the convergence of consistent finite difference approximations of well-posed initial-boundary value problems is called the Lax-Richtmyer equivalence theorem. For a proof we refer to Strikwerda (2004).

The finite difference (3.2.12a)-(3.2.12c) leads to a linear algebraic system. Hence, stability can be verified by arguments from numerical linear algebra. This will be illustrated by the subsequent example.

Example. We study the stability of the $\Theta$-scheme (3.2.10a)-(3.2.10c) applied to the heat equation (??)-(??) in $\Omega = (a, b)^2, a, b \in \mathbb{R}, a < b$. For $\Omega_h = \{x_1, \cdots, x_{n_h}\}^T$, we
denote by $u^{(j)}_h$ the vector $u^{(j)}_h = (u_h, k(x_1, t_j), \ldots, u_h, k(x_{n_h}, t_j))^T$, $0 \leq j \leq M + 1$. Then, (3.2.10a)-(3.2.10c) is equivalent to the linear algebraic system

$$\begin{equation}
( I_h + kB_h(\Theta)) u^{(j)}_h = (I_h - kC_h(\Theta)) u^{(j-1)}_h + b^{(j)}_h.
\end{equation}$$

Here, $B_h(\Theta) = \Theta A_h, C_h(\Theta) = (1 - \Theta) A_h$, where $A_h \in \mathbb{R}^{n_h \times n_h}$ refers to the block-tridiagonal matrix given by (2.2.19) and $b^{(j)}_h \in \mathbb{R}^{n_h}$. If $\delta u^{(0)}_h \in \mathbb{R}^{n_h}$ and $\delta b^{(j)}_h \in \mathbb{R}^{n_h}, 1 \leq j \leq M + 1$, are perturbations in the initial data and the right-hand side, respectively, and $z^{(j)}_h$ denotes the solution of (3.2.17) with respect to the perturbed data, by induction we can show

$$\begin{equation}
\|u^{(j)}_h - z^{(j)}_h\| \leq \|D_h(\Theta)\|^j \|\delta u^{(0)}_h\| + \\
+ \sum_{\ell=0}^{j-1} \|D_h(\Theta)\|^{j-\ell-1} \| (I_h + kB_h(\Theta))^{-1} \| \| \delta b^{(\ell)}_h \|,
\end{equation}$$

where $D_h(\Theta) := (I_h + kB_h(\Theta))^{-1}(I_h - kC_h(\Theta))$. Note that the norms stand for the Euclidean vector norm and the associated spectral norm of a matrix, respectively. It follows from (3.2.18) that for stability it suffices to show $\rho(D_h(\Theta)) < 1$ where $\rho(D_h(\Theta))$ is the spectral radius of $D_H(\Theta)$. Since we know the eigenvalues of $A_h$ (cf. (2.2.19)), the eigenvalues of $D_h(\Theta)$ can be assessed easily (Exercise). Under the constraint $k \leq h^2(2 - 4\Theta), \Theta < 1/2$, we deduce $\rho(D_h(\Theta)) < 1$, whereas $\rho(D_H(\Theta)) < 1, \Theta \geq 1/2$, without any constraints. Hence, the $\Theta$-scheme is conditionally stable for $\Theta < 1/2$ and unconditionally stable for $\Theta \geq 1/2$.


### 3.3 Weak solutions

We consider the following initial-boundary value problem for a parabolic differential equation

$$\begin{align}
(3.3.1a) \quad & \frac{\partial u}{\partial t} + A(t)u = f \quad \text{in } Q := \Omega \times (0, T), \\
(3.3.1b) \quad & u = 0 \quad \text{on } \Gamma \times (0, T), \\
(3.3.1c) \quad & u(\cdot, 0) = u_0 \quad \text{in } \Omega.
\end{align}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary $\Gamma = \partial \Omega$ and $T > 0$. We further assume $A(t), t \in (0, T)$, to be the formally self-adjoint second order linear elliptic differential operator

$$A(t)u(\cdot, t) := -\nabla \left( a(\cdot, t) \nabla u(\cdot, t) \right) + b(\cdot, t) \cdot \nabla u(\cdot, t) + c(\cdot, t) u(\cdot, t).$$

We suppose that $a = (a_{ij})_{i,j=1}^d, a_{ij} \in C[0, T], L^\infty(\Omega)), 1 \leq i, j \leq d$, and assume that for all $t \in (0, T)$ the matrix-valued function $(a_{ij}(t, \cdot))_{i,j=1}^d$ is uniformly positive definite in $\Omega$, i.e., there exists a constant $\alpha > 0$ such that
\[
\sum_{i,j=1}^{d} a_{ij}(t,x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d.
\]

Moreover, we suppose that \(b = (b_1, \cdots, b_d)^T\) such that \(b_i \in C([0, T], L^\infty(\Omega))\), \(1 \leq i \leq d\), and \(c \in C([0, T], L^\infty(\Omega))\) such that \(c(x, t) \geq 0, t \in [0, T]\), for almost all \(x \in \Omega\) as well as \(f \in L^2((0, T), H^{-1}(\Omega))\) and \(u_0 \in L^2(\Omega)\).

Following Renardy and Rogers (1993), we provide a weak solution concept for initial-boundary value problems associated with parabolic differential operators in a more general framework: Let \(H\) and \(V\) be Hilbert spaces with continuous and dense embeddings \(V \hookrightarrow H \hookrightarrow V^*\) where \(H\) is identified with its dual \(H^*\). We denote by \(\langle \cdot, \cdot \rangle_{V^*, V}\) the dual product of \(V^*\) and \(V\).

We refer to \(L^2((0, T); H)\) as the Hilbert space equipped with the norm
\[
\|u\|^2_{L^2((0, T); H)} := \int_0^T \|u(t)\|^2_H dt
\]
and define \(L^2((0, T); V)\) and its norm analogously. Moreover, we denote by \(H^1((0, T); V^*)\) the Hilbert space with the norm
\[
\|u\|^2_{H^1((0, T); V^*)} := \int_0^T \left( \|u(t)\|^2_{V^*} + \|u_t(t)\|^2_{V^*} \right) dt.
\]

(3.3.2) **Lemma:** The space \(L^2((0, T); V) \cap H^1((0, T); V^*)\) is continuously embedded in \(C([0, T]; H)\).

**Proof.** We assume \(u \in C^1([0, T]; H)\). For \(t \in [0, T]\), we choose \(t_0 \in [0, T]\) such that
\[
\|u(t_0)\|^2_H = T^{-1} \int_0^T \|u(t)\|^2_H dt = T^{-1}\|u\|^2_{L^2((0, T); H)}.
\]

It follows that
\[
\|u(t)\|^2_H = \|u(t_0)\|^2_H + 2 \int_{t_0}^t (u_s(s), u(s))_H ds \leq
\]
\[
\leq T^{-1}\|u\|^2_{L^2((0, T); H)} + 2 \int_0^T \|u_t\|_{V^*} \|u\|_V dt \leq
\]
\[
\leq T^{-1}\|u\|^2_{L^2((0, T); H)} + 2\|u\|_{H^1((0, T); V^*)} \|u\|_{L^2((0, T); V)}.
\]

We conclude using that \(C^1([0, T]; H)\) is dense in \(C([0, T]; H)\). \(\square\)
Now, let $A(t) : V \to V^*$, $t \in [0, T]$, be a bounded linear operator depending continuously on $t$. We assume that the bilinear form $a(t, \cdot, \cdot) : V \times V \to \mathbb{R}$ given by

(3.3.3) \[ a(t, u, v) := \langle A(t)u, v \rangle_{V^*, V} \text{ , } t \in [0, T] \]

satisfies the coercivity condition

(3.3.4) \[ a(t, v, v) \geq \alpha \|v\|_V^2 - \beta \|v\|_{H^1}^2 \text{ , } v \in V \text{ , } t \in [0, T] \]

where $\alpha$ and $\beta$ are positive constants independent of $t \in [0, T]$.

Given an initial condition $u^0 \in H$ and a right-hand side $f \in L^2((0, t), V^*)$, we consider the evolution equation

(3.3.5a) \[ u_t - A(t)u = f(t) , \]

(3.3.5b) \[ u(0) = u^0 . \]

(3.3.6) **Theorem:** Let $H, V$, $A(t) : V \to V^*$, $t \in [0, T]$, and $u_0, f$ be given as above. Then, the evolution equation (3.3.5a), (3.3.5b) admits a unique solution $u \in L^2((0, T), V) \cap H^1((0, T), V^*)$.

**Proof.** Without restriction of generality we may assume that $a(t, \cdot, \cdot)$ is uniformly $V$-elliptic, i.e., $\beta = 0$ in (3.3.4).

For the proof of the existence of a solution we use the technique of the Galerkin approximation, i.e., we approximate (3.3.5a), (3.3.5b) by a sequence $\{V_n\}_N$ of finite dimensional subspaces $V_n \subset V, n \in \mathbb{N}$, such that $V_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}$, where $\varphi_n, n \in \mathbb{N}$, are linearly independent elements in $V$ whose linear hull is dense in $V$. We denote by $P_n, n \in \mathbb{N}$, the orthogonal projection of $H$ onto $V_n$ and define $u_n \in C^1([0, T]; V_n)$ as the solution of the initial value problem

(3.3.7a) \[ (\frac{du_n}{dt}, \varphi_i)_H + a(t, u_n, \varphi_i) = \langle f(t), \varphi_i \rangle_{V^*, V} \text{ , } 1 \leq i \leq d , \]

(3.3.7b) \[ u_n(0) = P_n u^0 . \]

The ansatz $u_n(t) = \sum_{i=1}^n \kappa_i(t) \varphi_i$ shows that (3.3.7a), (3.3.7b) represents an initial value problem for a system of linear ordinary differential equations of first order in $(\kappa_1, \ldots, \kappa_n)^T$ which has a unique solution. Observing $\|P_n u^0\|_H \leq \|u^0\|_H, n \in \mathbb{N}$, we deduce the existence of a constant $C > 0$, independent of $n \in \mathbb{N}$, such that

(3.3.8) \[ \|u_n\|_{L^2((0, T); V)} \leq C \left( \|f\|_{L^2((0, T); V^*)} + \|u^0\|_H \right) , n \in \mathbb{N} . \]

Consequently, the sequence $\{u_n\}_N$ is bounded in $L^2((0, T); V)$. This implies the existence of a subsequence $\mathbb{N}' \subset \mathbb{N}$ and an element $u \in L^2((0, T); V)$ such that
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(3.3.9) \[ u_n \rightharpoonup u \quad (n \in \mathbb{N}', \ n \to \infty) \quad \text{in} \ L^2((0,T);V). \]

Now, suppose that \( \varphi_N \in C_0^\infty((0,T);V_N), N \in \mathbb{N} \) is given by

(3.3.10) \[ \varphi_N = \sum_{i=1}^{N} \gamma_i(t) \varphi_i, \]

where \( \gamma_i \in C_0^\infty((0,T)), 1 \leq i \leq d \). In view of (3.3.7a), for \( n \geq N \) there holds

\[ -(u_n, \frac{d\varphi_N}{dt})_H + a(t,u_N,\varphi_N) = \langle f(t), \varphi_N \rangle_{V^*,V}. \]

Since \( C_0^\infty((0,T);V_N) \) is dense in \( C_0^\infty((0,T);V) \), for all \( \varphi \in C_0^\infty((0,T);V) \) we thus have

\[ -(u_n, \frac{\partial \varphi}{\partial t})_H + a(t,u_N,\varphi) = \langle f(t), \varphi \rangle_{V^*,V}. \]

Due to (3.3.9), integration over \((0,T)\) and passing to the limit \( n \to \infty \) implies that for all \( \varphi \in C_0^\infty((0,T);V) \)

(3.3.11) \[ -\int_0^T (u_n, \frac{\partial \varphi}{\partial t})_H dt + \int_0^T a(t,u,\varphi) dt = \int_0^T \langle f(t), \varphi \rangle_{V^*,V} dt . \]

From (3.3.11) we deduce \( \partial u/\partial t \in L^2((0,T);V^*) \) and

(3.3.12) \[ \int_0^T (\frac{\partial u}{\partial t}, \varphi)_{V^*,V} dt = -\int_0^T (u_n, \frac{\partial \varphi}{\partial t})_H dt , \quad \varphi \in C_0^\infty((0,T);V). \]

Consequently, \( u \in H^1((0,T);V^*) \) whence \( u \in C([0,T];H) \) by Lemma 3.3.2. Using (3.3.12) in (3.3.11) it follows that \( u \) satisfies the evolution equation (3.3.5a).

It remains to be shown that \( u \) fulfills the initial condition \( u(0) = u^0 \in H \).

Let \( \varphi_N \) as in (3.3.10) with \( \gamma_i \in C^\infty([0,T]), \gamma_i(T) = 0, 1 \leq i \leq N \), such that for \( n \geq N \)

\[ -(u_n, \frac{d\varphi_N}{dt})_H + a(t,u_N,\varphi_N) = (u_n(0), \varphi_N(0))_H + \langle f(t), \varphi_N \rangle_{V^*,V}. \]

Since the linear space of such functions \( \varphi_N \) is dense in \( \{ \varphi \in H^1((0,T);V) \mid \varphi(T) = 0 \} \), the previous equation holds true for all such functions. Observing \( P_n u^0 \rightarrow u^0 \quad (n \to \infty) \) in \( H \), integration over \((0,T)\) and passing to the limit \( n \to \infty \) yields

(3.3.13) \[ \int_0^T (u_n, \frac{\partial \varphi}{\partial t})_H dt + \int_0^T a(t,u,\varphi) dt = \]

\[ = (u^0, \varphi(0))_H + \int_0^T \langle f(t), \varphi \rangle_{V^*,V} dt . \]
On the other hand, multiplication of (3.3.5a) by a function \( \varphi \in H^1((0,T);V) \), integration over \((0,T)\) and subsequent partial integration results in

\[
- \int_0^T (u, \frac{\partial \varphi}{\partial t})_H \, dt + \int_0^T a(t,u,\varphi) \, dt = \langle (u(0), \varphi(0))_H + \int (f(t), \varphi)_{V^*,V} \, dt .
\]

Inserting (3.3.14) into (3.3.13) gives \((u(0), \varphi(0))_H = (u^0, \varphi(0))_H\) which proves \(u(0) = u^0\).

As far as the uniqueness of a solution is concerned, we obtain from (3.3.3)

\[
\frac{1}{2} (\|u(t)\|^2_H - \|u(0)\|^2_H) + \int_0^t a(s;u,u) \, ds = \int_0^t (f,u)_H \, ds.
\]

Using the coercivity of the bilinear form results in the a priori estimate

\[
\|u\|_{L^2((0,T);V)} \leq C \left( \|f\|_{L^2((0,T);V^*)} + \|u^0\|_H \right).
\]

Now, if \(u_1, u_2 \in L^2((0,T);V) \cap H^1((0,T);V^*)\) are two solutions of the evolution equation, it follows that \(z := u_1 - u_2\) satisfies (3.3.5a),(??) with \(f = 0\) and \(z^0 = 0\). The estimate (3.3.15) shows \(z = 0\).

\[\square\]

### 3.4 Finite element methods

Let \(V \subseteq H^1(\Omega)\) be a Hilbert space and let \(a(t,\cdot,\cdot) : V \times V \to \mathbb{R}\) be a bilinear form that is uniformly coercive \(t \in [0,T], T \in \mathbb{R}_+\). Further, assume \(u^0 \in L^2(\Omega)\) and \(f \in L^2((0,T);V^*)\). According to Theorem (??), the evolution equation

\[
(3.4.1a) \quad (u_t,v)_{0,\Omega} + a(t,u,v) = \langle f(t), v \rangle_{V^*,V} , \; v \in V ,
\]

\[
(3.4.1b) \quad (u(0),v)_{0,\Omega} = (u^0,v)_{0,\Omega} , \; v \in V ,
\]

admits a unique solution \(u \in L^2((0,T);V) \cap H^1((0,T);V^*)\).

The standard finite element method for the numerical solution of this evolution equation consists in the discretization in space by restricting (3.4.1a),(3.4.1b) to a finite element space \(V_h \subset V, \dim V_h = n_h\). This leads to the semidiscrete approximation

\[
(3.4.2a) \quad ((u_h)_t,v_h)_{0,\Omega} + a(t,u_h,v_h) = \langle f(t), v_h \rangle_{V^*,V} , \; v_h \in V_h ,
\]

\[
(3.4.2b) \quad (u_h(0),v_h)_{0,\Omega} = (u^0,v_h)_{0,\Omega} , \; v_h \in V_h .
\]
As we already know from the proof of Theorem (??), the equations (3.4.2a), (3.4.2b) represent an initial value problem for a system of linear ordinary differential equations which admits a unique solution: If \( (\varphi_h^{(i)})_{i=1}^{n_h} \) is a basis of \( V_h \), we can write the solution \( u_h(t) \in V_h, t \in [0, T] \), according to \( u_h(t) = \sum_{j=1}^{n_h} \kappa_j(t)\varphi_h^{(j)} \) and thus obtain

\[
\begin{align*}
(3.4.3a) & \quad M_h \kappa'(t) + A_h(t) \kappa(t) = b_h(t) \quad t \in [0, T], \\
(3.4.3b) & \quad \kappa(0) = \kappa_0.
\end{align*}
\]

Here, \( M_h \in \mathbb{R}^{n_h \times n_h} \) stands for the mass matrix

\[
M_h := \begin{pmatrix}
(\varphi_h^{(1)}, \varphi_h^{(1)})_{0, \Omega} & \cdots & (\varphi_h^{(n_h)}, \varphi_h^{(1)})_{0, \Omega} \\
\vdots & \ddots & \vdots \\
(\varphi_h^{(1)}, \varphi_h^{(n_h)})_{0, \Omega} & \cdots & (\varphi_h^{(n_h)}, \varphi_h^{(n_h)})_{0, \Omega}
\end{pmatrix}
\]

and \( A_h(t) \in \mathbb{R}^{n_h \times n_h}, t \in [0, T], \) denotes the stiffness matrix

\[
A_h(t) := \begin{pmatrix}
a(t, \varphi_h^{(1)}, \varphi_h^{(1)}) & \cdots & a(t, \varphi_h^{(n_h)}, \varphi_h^{(1)}) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{n_h} a(t, \varphi_h^{(1)}, \varphi_h^{(i)}) & \cdots & \sum_{i=1}^{n_h} a(t, \varphi_h^{(n_h)}, \varphi_h^{(i)})
\end{pmatrix},
\]

whereas the vectors \( b_h(t) \in \mathbb{R}^{n_h}, t \in [0, T], \) and \( \kappa_0 \in \mathbb{R}^{n_h} \) are given by

\[
b_h(t) := (\langle f(t), \varphi_h^{(1)} \rangle_{V^*, V}, \cdots, \langle f(t), \varphi_h^{(n_h)} \rangle_{V^*, V})^T, \\
\kappa_0 = ((u_0^0, \varphi_h^{(1)}), 0, \Omega), \cdots, (u_0^0, \varphi_h^{(n_h)}), 0, \Omega)^T.
\]

For the discretization in time with respect to a partition \( \tilde{I}_k \) of the time interval \( [0, T] \) according to (??), we use an approximation of the time derivative by the forward or backward difference quotient or a convex combination of both as in the \( \Theta \)-method described in chapter ??, as the solution of

\[
\begin{align*}
(3.4.4a) & \quad k^{-1} M_h (u_{h, k+1} - u_{h, k}) + \Theta A_h(t_{k+1}) u_{h, k+1} + \\
& \quad \quad + (1 - \Theta) A_h(t_k) u_{h, k} = b_\Theta^0(t_k, t_{k+1}), \quad 0 \leq k \leq M, \\
(3.4.4b) & \quad u_{h, 0} = u_h^0,
\end{align*}
\]

where \( b_\Theta^0(t_k, t_{k+1}) := \Theta b_h(t_{k+1}) + (1 - \Theta) b_h(t_k) \) and \( u_h^0 = \sum_{j=1}^{n_h} \kappa_j^0 \varphi_h^{(j)} \).

We derive a priori estimates of the global discretization error \( u(t) - u_{h, k}(t), t \in \tilde{I}_k \), in the special case of the linear heat equation, i.e., \( A(t) = A = -\Delta \) in (3.3.1a). For the general case we refer to Thomée (1997). We assume that the finite element spaces \( V_h \) satisfy the following approximation property: For \( v \in H^{r+1}(\Omega) \) there holds
\[ \inf_{v_h \in V_h} \left( \| v - v_h \|_{0,\Omega} + h \| v - v_h \|_{1,\Omega} \right) \leq Ch^{r+1}, \]

For the backward difference quotient in time, i.e., \( \Theta = 1 \) in (3.4.4a),(3.4.4b), we first consider an estimate in the \( H^1 \)-seminorm.

**Theorem**: Let \( u \in H^1(0,T); H^{r+1}(\Omega) \cap H^2((0,T); L^2(\Omega)) \) be the solution of the initial-boundary value problem (3.3.1a)-(3.3.1c) with \( A(t) = A = -\Delta \) and let \( u_{h,k} \) be the solution of (3.4.4a),(3.4.4b) in case \( \Theta = 1 \). Then, there exists a constant \( C > 0 \), independent of \( h \) and \( k \), such that for all \( 0 \leq m \leq M + 1 \)

\[ |u(t_m) - u_{h,k}(t_m)|_{1,\Omega} \leq C(h^r + k). \]

**Proof**. We set \( e(t_m) := u(t_m) - u_{h,k}(t_m) \). Denoting by \( P_h^a : V \to V_h \) the elliptic projection onto \( V_h \) (cf. chapter ???), the error can be split according to \( e(t_m) = e_1(t_m) + e_2(t_m) \) mit

\[ (3.4.6) \quad e_1(t_m) := u(t_m) - P_h^a u(t_m), \quad e_2(t_m) := P_h^a u(t_m) - u_{h,k}(t_m). \]

Taking advantage of Theorem (??), \( e_1(t_m) \) can be estimated from above as follows

\[ (3.4.7) \quad |e_1(t_m)|_{1,\Omega} \leq C h^r \| u(t_m) \|_{r+1,\Omega} \leq C h^r \left( \| u^0 \|_{r+1,\Omega} + \int_0^{t_m} \| u(t) \|_{r+1,\Omega} \, dt \right). \]

On the other hand, \( e_2(t_m) \) satisfies the variational equation

\[ (3.4.8) \quad (D_k e_2(t_m), v_h)_{0,\Omega} + (\nabla e_2(t_m), \nabla v_h)_{0,\Omega} = (g(t_m), v_h), \quad v_h \in V_h, \]

where \( g(t_m) := P_h^a D_k u(t_m) - u_t(t_m) \). If we choose \( v_h = D_k^- e_2(t_m) \) in (3.4.8), we obtain

\[ k \| D_k^- e_2(t_m) \|^2_{0,\Omega} + (\nabla e_2(t_m), \nabla e_2(t_m))_{0,\Omega} = (e_2(t_m), e_2(t_{m-1})) + k (g(t_m), D_k^- e_2(t_m))_{0,\Omega}, \]

whence

\[ |e_2(t_m)|_{1,\Omega}^2 \leq C \left( |e_2(t_{m-1})|_{1,\Omega}^2 + k \| g(t_m) \|_{0,\Omega}^2 \right). \]

A repeated application of this reasoning results in

\[ (3.4.9) \quad |e_2(t_m)|_{1,\Omega}^2 \leq C \left( |e_2(0)|_{1,\Omega}^2 + k \sum_{\ell=1}^m \| g(t_\ell) \|_{0,\Omega}^2 \right). \]

In view of the assumption \( \| u^0 - u_{h,k}^0 \|_{1,\Omega} \leq Ch^r \), for the first term on the right-hand side in (3.4.9) it follows that
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\begin{equation}
|e_2(0)|_{1, \Omega} \leq |u^0 - u^0_{h,k}|_{1, \Omega} + |u^0 - P^a u^0|_{1, \Omega} \leq Ch^r.
\end{equation}

In order to estimate the second term, we split \( g(t_\ell) \) according to
\[
g_1(t_\ell) := k^{-1} \int_{t_{\ell-1}}^{t_\ell} (P^a_h - I) u_t(t) \, dt,
\]
\[
g_2(t_\ell) := k^{-1} (u(t_\ell) - u(t_{\ell-1}) - ku_{t}(t_\ell)).
\]

Another application of Theorem (??) yields
\[
k \| g_1(t_\ell) \|_{0, \Omega}^2 \leq C h^r \int_{t_{\ell-1}}^{t_\ell} \| u_t(t) \|_{r+1, \Omega}^2 \, dt.
\]

Taking advantage of
\[
g_2(t_\ell) = -k^{-1} \int_{t_{\ell-1}}^{t_\ell} (t - t_{\ell-1}) u_{tt}(t) \, dt,
\]
we get
\[
k \| g_2(t_\ell) \|_{0, \Omega}^2 \leq C k^2 \int_{t_{\ell-1}}^{t_\ell} \| u_{tt}(t) \|_{0, \Omega}^2 \, dt.
\]

If we sum over all \( \ell \), the two previous estimates give
\begin{equation}
\sum_{\ell=1}^{m} \| g(t_\ell) \|_{0, \Omega}^2 \leq \leq C \left( h^{2r} \int_{0}^{t_m} \| u_t(t) \|_{r+1, \Omega}^2 \, dt + k^2 \int_{0}^{t_m} \| u_{tt}(t) \|_{0, \Omega}^2 \, dt \right).
\end{equation}

The assertion now follows by using (3.4.10),(3.4.11) in (3.4.9) as well as in view of (3.4.7).

Under stronger regularity assumptions on the classical solution of the initial-boundary value problem, for the Crank-Nicolson-Galerkin method, i.e., \( \Theta = 1/2 \), the order of convergence 2 in \( k \) can be established.

\begin{equation}
\textbf{Theorem: Let}
\end{equation}
\[
u \in H^1((0, T); H^{r+1}(\Omega)) \cap H^2((0, T); H^2(\Omega)), \quad \Delta u_{tt} \in L^2((0, T); L^2(\Omega))
\]
be the solution of the initial-boundary value problem (3.3.1a)-(3.3.1c) in case \( A(t) = A, = -\Delta \) and let \( u_{h,k} \) be the solution of (3.4.4a),(3.4.4b) for \( \Theta = 1/2 \).
Then, there exists a constant $C > = $, independent of $h$ and $k$, such that for all $0 \leq m \leq M + 1$

$$|u(t_m) - u_{h,k}(t_m)|_{1,\Omega} \leq C(h^r + k^2).$$

**Proof.** We split the error $e(t_m) = u(t_m) - u_{h,k}(t_m)$ as in (3.4.6) and estimate $e_1(t_m)$ from above as in (3.4.7). The second error term $e_2(t_m)$ satisfies the variational equation

$$(D_k - k e_2(t_m), v_h)_{0,\Omega} + (\nabla e_2(t_m) + e_2(t_{m-1}))/2, \nabla v_h)_{0,\Omega} = (g(t_m), v_h)_{0,\Omega}, \quad v_h \in V_h,$$

where $g(t_m) := P_h^a D^- k (u(t_m) + u(t_{m-1}))/2$. If we choose $v_h := (e_2(t_m) + e_2(t_{m-1}))/2$ and split $g(t_m)$ according to

$$g(t_m) = (P - h^a - I)D_k^+ u(t_m) + (D_k^- u(t_m) - u_t(t_m - k/2)) +$$
$$- \Delta(u(t_m - k/2) - (u(t_m) + u(t_{m-1}))/2)$$

we can estimate $e_2(t_m)$ from above as in the proof of Theorem (3.4.5). \qed

From Theorem (??) we know that for $(r + 1)$-regular elliptic boundary value problems we can achieve an optimal a priori estimate in the $L^2$-norm, i.e., compared to the a priori estimate in the $H^1$-norm we gain a power in $h$. This result can be obtained for the linear heat equation as well.

(3.4.14) **Theorem:** Assume that the conditions of Theorem (??) and Theorem (??) hold true and that the boundary value problem for the Poisson equation is $(r + 1)$-regular. Then, there exists a constant $C > = $, independent of $h$ and $k$, such that for all $0 \leq m \leq M + 1$

$$\|u(t_m) - u_{h,k}(t_m)\|_{0,\Omega} \leq C(h^{r+1} + k^p),$$

where $p = 1$ for $\Theta = 1$ and $p = 2$ for $\Theta = 1/2$.

**Proof.** The proof can be accomplished as in Theorem (??) and as in Theorem (??), if we choose $v_h = e_2(t_m)$ in (3.4.8) and $v_h = (e_2(t_m) + e_2(t_{m-1}))/2$ in (3.4.13). \qed

**Remark** (10.4.4.15) As far as a priori estimates in the $L^\infty$-norm are concerned, we refer to Thomée (1997).
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