Numerical Partial Differential Equations
(Homework 2)

Exercise 2 (Compact Finite Difference Approximation of the Laplacian)
Show that it is not possible to obtain a compact finite difference approximation of the Laplacian of order \( O(h^3) \).

4 Points

Exercise 3 (Higher Order Finite Difference Approximation)
Let \( \Omega := (a, b)^2, a, b \in \mathbb{R}, a < b \), and assume that \( \Omega_h \) is a uniform grid of step size \( h > 0 \). For the approximation of Poisson’s equation with homogeneous Dirichlet boundary conditions consider the finite difference approximation

\[
- \frac{1}{6} (u_h(x_1 + h, x_2 + h) + u_h(x_1 + h, x_2 - h) + u_h(x_1 - h, x_2 + h) + \\
+ u_h(x_1 - h, x_2 - h)) + \frac{2}{3} (u_h(x_1 + h, x_2) + u_h(x_1 - h, x_2) + \\
+ u_h(x_1, x_2 + h) + u_h(x_1, x_2 - h)) + \frac{10}{3} u_h(x_1, x_2) = \\
\frac{h^2}{12} (f(x_1 + h, x_2) + f(x_1 - h, x_2) + f(x_1, x_2 + h) + \\
+ f(x_1, x_2 - h) + 8f(x_1, x_2)) , \quad (x_1, x_2) \in \Omega_h , \\
u_h(x) = 0 , \quad (x_1, x_2) \in \Gamma_h ,
\]

and show that for \( u \in C^6(\Omega) \) it is consistent of order \( O(h^4) \).

4 Points
**Exercise 4** *(Comparison of Finite Difference Approximations)*

For the finite difference approximation of Poisson’s equation

\[ -\Delta u = f \quad \text{in } \Omega = (0,1)^2 , \]
\[ u = 0 \quad \text{on } \Gamma = \partial \Omega , \]

with respect to uniform grids \( \Omega_{h_\nu} \) of step sizes \( h_\nu = 1/(N_\nu + 1) \) use

(i) the five-point difference approximation of the Laplacian,
(ii) the nine-point difference approximation of the Laplacian,
(iii) the finite difference approximation from Exercise 3,

in case \( f \) is chosen such that \( u(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2) \) is the exact solution.

For \( h_\nu = 2^{-(\nu+2)}, 1 \leq \nu \leq 4 \), compute

\[ e_\nu := \max_{x \in \Omega_h} |u(x) - u_h(x)| , \quad 1 \leq \nu \leq 4 , \]

and

\[ r_\nu := \frac{\log(e_{\nu-1}/e_\nu)}{\log(h_{\nu-1}/h_\nu)} , \quad 2 \leq \nu \leq 4 . \]

Show the results in form of a table

<table>
<thead>
<tr>
<th>( h_\nu )</th>
<th>Five-Point</th>
<th>Nine-Point</th>
<th>FD from Ex. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{8} )</td>
<td>( e_\nu )</td>
<td>( r_\nu )</td>
<td>( e_\nu )</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td></td>
<td></td>
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<tr>
<td>( \frac{1}{32} )</td>
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<tr>
<td>( \frac{1}{64} )</td>
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</tr>
</tbody>
</table>

Display \( e_\nu \) in the form \( x.xxx \times 10^{-xx} \) and \( r_\nu \) in the form \( x.xxx \).

6 Points

**Exercises 2 and 3 are due on February 6, 2008. Exercise 4 is due on February 13, 2008. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.**