Exercise 6 (The notorious L-shaped domain)

Let $\Omega \subset \mathbb{R}^2$ be the L-shaped domain

$$\Omega := (-1,0) \times (-1,1) \cup [0,1) \times (0,1)$$

and consider the Laplace equation

$$-\Delta u = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_D,$$
$$\nu \cdot \nabla u = g \quad \text{on } \Gamma_N,$$

with homogeneous Dirichlet boundary conditions on

$$\Gamma_D := \{0\} \times [-1,0] \cup [0,1] \times \{0\}$$

and inhomogeneous Neumann boundary conditions on

$$\Gamma_N := \partial \Omega \setminus \Gamma_D.$$

The inhomogeneous Neumann boundary data $g$ are chosen such that

$$u(r,\varphi) = r^{2/3} \sin(2\varphi/3)$$

is the exact solution of the problem (in polar coordinates).

Show that $u \in H^1(\Omega)$, but $u \notin H^2(\Omega)$.

6 Points

Remark: The solution is in $H^{1+2/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0.$
**Exercise 7** *(Geometric representation of nodal basis functions)*

Let $T$ be a nondegenerate triangle in $\mathbb{R}^2$ with vertices $a_i$ and edges $e_i$, $1 \leq i \leq 3$, where the edge $e_i$ is opposite to the vertex $a_i$. Denote by $m_{e_i}$ the midpoints and by $\nu_{e_i}$ the exterior unit normal vectors associated with the edges. Show that the nodal basis functions $\varphi_i$ for the finite element space of continuous, piecewise linear finite elements admit the representation

$$
\varphi_i(x) = \frac{|e_i|}{2|T|} \nu_{e_i} \cdot (x - m_{e_i}), \quad 1 \leq i \leq 3,
$$

where $|e_i|$ is the length of $e_i$ and $|T|$ stands for the area of $T$.

6 Points

The exercises are due on March 12, 2008. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.