Chapter 3 Conforming Finite Element Methods

3.1 Foundations

3.1.1 Ritz-Galerkin Method

Let V be a Hilbert space, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ a bounded, V-elliptic bilinear form and $\ell : V \to \mathbb{R}$ a bounded linear functional. We want to approximate the **variational equation**:
Find $u \in V$ such that

$$a(u,v) \;=\; \ell(v) \quad , \quad v \in V \; . \tag{3.1}$$

We recall that the **Lax-Milgram Lemma** ensures the existence and uniqueness of a solution of (3.1).
Now, given a finite-dimensional subspace $V_h \subset V$, $\dim\, V_h = n_h$, the **Ritz-Galerkin method** is to approximate (3.1) by its restriction to V_h:
Find $u_h \in V_h$ such that

$$a(u_h,v_h) \;=\; \ell(v_h) \quad , \quad v_h \in V_h \; . \tag{3.2}$$

Again, by the Lax-Milgram Lemma we know that (3.2) has a unique solution $u_h \in V_h$.
It is easy to see that the Ritz-Galerkin method gives rise to a **linear algebraic system**, once we specify a **basis** of V_h. Therefore, let us assume that $(\varphi_h^{(i)})_{i=1}^{n_h}$ is a basis of V_h, i.e.,

$$V_h \;=\; \text{span}(\varphi_h^{(1)},...,\varphi_h^{(n_h)}) \; . \tag{3.3}$$

Then, the solution $u_h \in V_h$ of (3.2) can be represented as a linear combination of the basis functions according to

$$u_h \;=\; \sum_{j=1}^{n_h} \alpha_j \; \varphi_h^{(j)} \; . \tag{3.4}$$

Apparently, u_h as given by (3.4) satisfies (3.2) if and only if

$$\sum_{j=1}^{n_h} a(\varphi_h^{(j)},\varphi_h^{(i)}) \; \alpha_j \;=\; \ell(\varphi_h^{(i)}) \quad , \quad 1 \le i \le n_h \; . \tag{3.5}$$

Obviously, (3.5) represents a **linear algebraic system**

$$A_h \; \alpha_h \;=\; b_h \tag{3.6}$$

in the unknown vector $\alpha_h = (\alpha_1, ..., \alpha_{n_h})^T$, where the **stiffness matrix** $A_h \in \mathbb{R}^{n_h \times n_h}$ and the **load vector** $b_h \in \mathbb{R}^{n_h}$ are given by

$$A_h := \begin{pmatrix} a(\varphi_h^{(1)}, \varphi_h^{(1)}) & \cdot & \cdot & \cdot & a(\varphi_h^{(n_h)}, \varphi_h^{(1)}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a(\varphi_h^{(1)}, \varphi_h^{(n_h)}) & \cdot & \cdot & \cdot & a(\varphi_h^{(n_h)}, \varphi_h^{(n_h)}) \end{pmatrix} \tag{3.7}$$

and

$$b_h^{(i)} := \ell(\varphi_h^{(i)}) \quad , \quad 1 \le i \le n_h \ . \tag{3.8}$$

There are basically **two major aspects** with regard to the construction of appropriate finite-dimensional subspaces V_h:

- The **efficient numerical solution** of the linear algebraic system (3.6).
- The **accuracy** of the approximation of the solution $u \in V$ of (3.1) by the solution $u_h \in V_h$ of (3.2).

3.1.2 Céa's Lemma

Céa's Lemma tells us that under the assumptions of the Lax-Milgram Lemma the accuracy of the solution $u \in V$ of (3.1) by the solution $u_h \in V_h$ of (3.2) is as good as the **best approximation** of $u \in V$ by a function in V_h which reduces this issue to a problem of approximation theory.
It is based on the observation that the error $u - u_h$ is **a-orthogonal** to V_h, i.e.,

$$a(u - u_h, v_h) \ = \ 0 \quad , \quad v_h \in V_h \ , \tag{3.9}$$

a property which is referred to as **Galerkin orthogonality**.

Definition 3.1 Elliptic projection

If the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic, it defines an inner product on V, the energy inner product. Then, (3.9) states that the solution $u_h \in V_h$ of (3.2) is the projection of the solution $u \in V$ of (3.1) onto V_h which is called the **elliptic projection**.

Lemma 3.1 Céa's Lemma

Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bounded, V-elliptic bilinear form, i.e., there exist positive constants C and α such that

$$|a(u,v)| \ \le \ C \ \|u\|_V \ \|v\|_V \quad , \quad u, v \in V \ , \tag{3.10}$$

$$a(u,u) \ \ge \ \alpha \ \|u\|_V^2 \quad , \quad u \in V \ . \tag{3.11}$$

Assume further that $\ell \in V^*$ is a bounded linear functional and that $u \in V$ and $u_h \in V_h$ are the unique solutions of (3.1) and (3.2), respectively.

Then, there holds

$$\|u - u_h\|_V \ \leq \ \frac{C}{\alpha} \ \inf_{v_h \in V_h} \|u - v_h\|_V \ . \tag{3.12}$$

Proof. Using the V-ellipticity and boundedness of $a(\cdot,\cdot)$ as well as the Galerkin orthogonality (3.9), we find that for any $v_h \in V_h$

$$\begin{aligned} \alpha \ \|u - u_h\|_V^2 \ &\leq \ a(u - u_h, u - u_h) \ = \\ &= \ a(u - u_h, u - v_h) \ + \ \underbrace{a(u - u_h, v_h - u_h)}_{= \ 0} \ \leq \\ &\leq \ C \ \|u - u_h\|_V \ \|u - v_h\|_V \ , \end{aligned}$$

from which we readily deduce (3.12). •

3.2. Triangulations

The construction of conforming finite element spaces is based on a suitable partition of the computational domain.

Definition 3.2 Triangulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. A **triangulation** $\mathcal{T}_h$ of $\overline{\Omega}$ is a partition of $\overline{\Omega}$ into a finite number of subsets K, called **finite elements**, such that

($\mathcal{T}_h$1) $\quad \overline{\Omega} \ = \ \bigcup_{K \in \mathcal{T}_h} K \ ,$

($\mathcal{T}_h$2) $\quad K \ = \ \overline{K} \quad , \quad K^o \ \neq \ \emptyset \quad , \quad K \in \mathcal{T}_h \ ,$

($\mathcal{T}_h$3) $\quad K_1^o \ \cap \ K_2^o \ = \ \emptyset$ for all $K_1, K_2 \in \mathcal{T}_h \ , \ K_1 \ \neq K_2 \ ,$

($\mathcal{T}_h$4) $\quad \partial K \ , \ K \in \mathcal{T}_h$ is Lipschitz continuous.

We consider two types of elements: **d-simplices** and **d-rectangles**.

Definition 3.3 d-simplex

A **d-simplex** K in $\mathbb{R}^d$ is the convex hull of $d+1$ points $a_j = (a_{ij})_{l=1}^d \in \mathbb{R}^d$:

$$K \ = \ \{ \ x = \sum_{j=1}^{d+1} \lambda_j \ a_j \ | \ 0 \leq \lambda_j \leq 1 \ , \ \sum_{j=1}^{d+1} \lambda_j = 1 \ \} \ .$$

The simplex K is called **non degenerate**, if any point $x \in \mathbb{R}^d$ can be uniquely represented in the form

$$x = \sum_{j=1}^{d+1} \lambda_j \ a_j \quad , \quad \lambda_j \in \mathbb{R} \ , \ \sum_{j=1}^{d+1} \lambda_j = 1 \ .$$

Remark 3.1 Non degeneracy of a d-simplex

The non degeneracy of a d-simplex is related to the unique solvability of the linear system

$$\underbrace{\begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1,d+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{d1} & \cdot & \cdot & \cdot & a_{d,d+1} \\ 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix}}_{=:\ A} \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \lambda_d \\ \lambda_{d+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_d \\ 1 \end{pmatrix} . \tag{3.13}$$

Obviously, K is non degenerate if and only if the matrix A is regular.

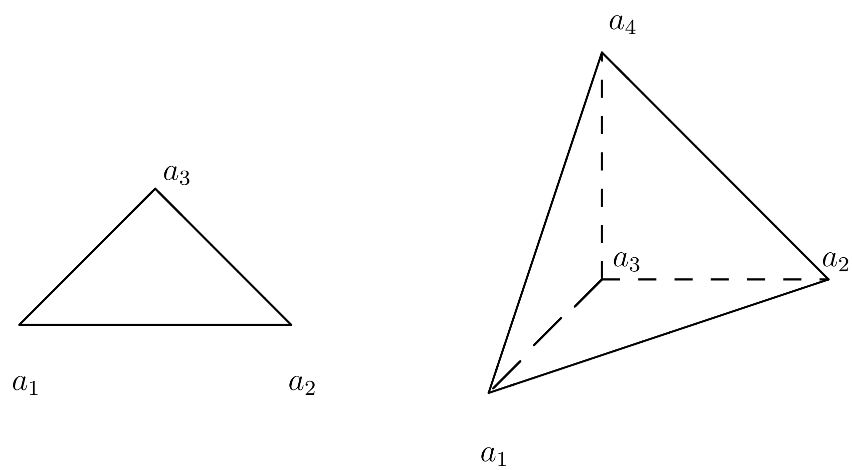

Figure 3.1: Triangle (d=2) and tetrahedron (d=3)

Definition 3.3 Vertices, edges, and faces

The points $a_j, 1 \leq j \leq d+1,$ of a d-simplex K are called **vertices**.
An **m-dimensional face** of a d-simplex $K, 0 \leq m \leq d,$ is an m-simplex whose vertices correspond to vertices of K.
A 1-dimensional face is referred to as an **edge**.
For $D \subseteq \overline{\Omega}$, we refer to $\mathcal{V}_h(D), \mathcal{F}_h(D),$ and $\mathcal{E}_h(D)$ as the sets of vertices, $(d-1)$-dimensional faces and edges of $\mathcal{T}_h$, respectively.
The d-simplex $\hat{K}$ with vertices $\hat{a}_1 = (0, ..., 0)^d$ and $\hat{a}_{i+1} = e_i, 1 \leq i \leq d,$ is referred to as the **unit d-simplex**.

Definition 3.4 Barycentric coordinates, center of gravity

The **barycentric coordinates** $\lambda_j, 1 \leq j \leq d+1$, of a point $x \in \mathbb{R}^d$ with respect to the $d+1$ vertices a_j of a non degenerate d-simplex K are the components of the unique solution of the linear system (3.13). The center of gravity x_S of a non degenerate d-simplex K is the point with

$$\lambda_j(x_S) \;=\; \frac{1}{d+1} \quad , \quad 1 \leq j \leq d+1 \; .$$

Lemma 3.2 Affine transformation

Any non degenerate d-simplex $K \subset \mathbb{R}^d$ is the image of the unit d-simplex $\hat{K}$ under an **affine transformation**

$$\begin{aligned} F_K \;:\; \mathbb{R}^d \quad &\rightarrow \quad \mathbb{R}^d \\ \hat{x} \quad &\longmapsto \quad F_K(\hat{x}) \;=\; B_K\hat{x} + b_K \end{aligned}$$

with a nonsingular matrix $B_K \in \mathbb{R}^{d\times d}$ and $b_K \in \mathbb{R}^d$.

Definition 3.5 Simplicial triangulation

A triangulation $\mathcal{T}_h$ of a polyhedral domain $\Omega \subset \mathbb{R}^d$ is called a **simplicial triangulation**, if its elements K are d-simplices.

Definition 3.6 d-rectangle

A **d-rectangle** K in $\mathbb{R}^d$ is the tensor product of d intervals $[c_i, d_i], c_i \leq d_i, 1 \leq i \leq d$, i.e.,

$$K \;=\; \prod_{i=1}^{d}[c_i, d_i] \;=\; \{\; x = (x_1, ..., x_d)^T \mid c_i \leq x_i \leq d_i \;,\; 1 \leq i \leq d \;\} \; .$$

A d-rectangle K is said to be **non degenerate**, if $c_i < d_i, 1 \leq i \leq d$. The d-rectangle $\hat{K} := [0,1]^d$ is called the unit d-rectangle (unit cube) in $\mathbb{R}^d$.

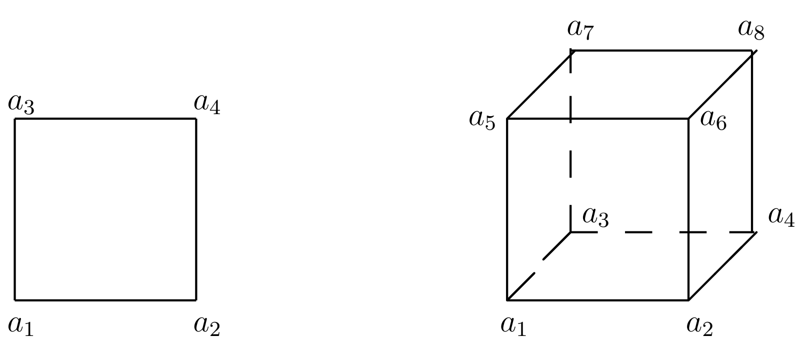

Figure 3.2: 2-Rectangle (d=2) and 3-rectangle (d=3)

Definition 3.7 Vertices, edges, and faces

The points $a_j, 1 \le j \le 2^d$, of a d-rectangle K given by

$$a_j = (a_{j_1}, ..., a_{j_d})^T \ , \ a_{j_i} = c_i \text{ or } a_{j_i} = d_i \ , \ 1 \le i \le d,$$

are called **vertices**.
An **m-dimensional face** of a d-rectangle $K, 1 \le m \le d-1,$ is an m-rectangle whose vertices correspond to vertices of K.
A 1-dimensional face is referred to as an **edge**.
For $D \subseteq \overline{\Omega}$, we refer to $\mathcal{V}_h(D), \mathcal{F}_h(D),$ and $\mathcal{E}_h(D)$ as the sets of vertices, $(d-1)$-dimensional faces and edges of $\mathcal{T}_h$, respectively.

Lemma 3.3 Diagonal affine transformation

Any non degenerate d-rectangle $K \subset \mathbb{R}^d$ is the image of the unit d-rectangle (unit cube) $\hat{K}$ under a **diagonal affine transformation**

$$\begin{aligned} F_K \ : \ \mathbb{R}^d & \quad \rightarrow \quad \mathbb{R}^d \\ \hat{x} & \quad \longmapsto \quad F_K(\hat{x}) \ = \ B_K \hat{x} + b_K \end{aligned}$$

with a nonsingular diagonal matrix $B_K = (b_{ii}^K)_{i=1}^d$ and $b_K \in \mathbb{R}^d$.

Definition 3.8 Rectangular triangulation

A triangulation $\mathcal{T}_h$ of a rectangular domain $\Omega \subset \mathbb{R}^d$ is called a **rectangular triangulation**, if its elements K are d-rectangles.

3.3 Local specification of finite elements

With respect to a triangulation $\mathcal{T}_h$ of the computational domain $\Omega \subset \mathbb{R}^d$, conforming finite element functions are defined locally for the elements $K \in \mathcal{T}_h$ and composed in such a way that the resulting globally defined function belongs to the underlying function space V.

Definition 3.9 Local trial functions and degrees of freedom

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$ and $K \in \mathcal{T}_h$. Assume that P_K is a linear space of functions $p : K \rightarrow \mathbb{R}$ with $P_K \subset H^1(K)$ and $\dim P_K = n_K$. the elements of P_K are called **local trial functions**.
Let $\ell_i \in P_K', 1 \le i \le n_K$ be bounded linear functionals $\ell_i : P_K \rightarrow \mathbb{R}, 1 \le i \le n_K$, and consider

$$\Sigma_K \ := \ \{ \ \ell_i(p) \mid p \in P_K \ , \ 1 \le i \le n_K \ \} \ . \tag{3.14}$$

The elements of Σ_K are called **degrees of freedom**.

Definition 3.10 Finite elements and unisolvence
Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$, $K \in \mathcal{T}_h$ and P_K, Σ_K as in Definition 3.9. Then, the triple (K, P_K, Σ_K) is called a **finite element**.
A finite element (K, P_K, Σ_K) is said to be **unisolvent**, if any $p \in P_K$ is uniquely determined by its degrees of freedom in Σ_K.

Definition 3.11 Affine equivalence of finite elements
Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$, $K \in \mathcal{T}_h$ and P_K, Σ_K as in Definition 3.9. Let further $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$ be a **reference element**. Then, the finite elements (K, P_K, Σ_K), $K \in \mathcal{T}_h$, are said to be **affine equivalent** to the reference element $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$, if there exists an invertible affine mapping $F_K : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $K \in \mathcal{T}_h$

$$K \;=\; F_K(\hat{K}) \;, \tag{3.15}$$

$$P_K \;=\; \{p : K \to \mathbb{R} \mid p = \hat{p} \circ F_K^{-1} \;,\; \hat{p} \in \hat{P}_{\hat{K}}\} \;, \tag{3.16}$$

$$\Sigma_K \;=\; \{\ell_i : P_K \to \mathbb{R} \mid \ell_i = \hat{\ell}_i \circ F_K^{-1} \;,\; \hat{\ell}_i \in \hat{\Sigma}_{\hat{K}} \;,\; 1 \le i \le n_K\} \;. \tag{3.17}$$

Definition 3.12 Finite element space
Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$ and $P_K, K \in \mathcal{T}_h$ as in Definition 3.9. Then

$$V_h \;:=\; \{\; v_h : \overline{\Omega} \to \mathbb{R} \mid v_h|_K \in P_K \;,\; K \in \mathcal{T}_h \;\} \tag{3.18}$$

is called a **finite element space**.

Theorem 3.1 H^1-conformity of finite elements
Let V_h be a finite element space and assume that

$$P_K \subset H^1(K) \quad , \quad K \in \mathcal{T}_h \;, \tag{3.19}$$

$$V_h \subset C^0(\Omega) \;. \tag{3.20}$$

Then

$$V_h \subset H^1(\Omega) \;. \tag{3.21}$$

Proof. Let $v_h \in V_h$. Obviously, $v_h \in L^2(\Omega)$. We have to show that v_h admits weak first derivatives $w_h^\alpha \in L^2(\Omega), |\alpha| = 1$, i.e.,

$$\int_\Omega v_h \; D^\alpha z \; dx \;=\; (-1)^{|\alpha|} \int_\Omega w_h^\alpha \; z \; dx \quad , \quad z \in C_0^\infty(\Omega) \;. \tag{3.22}$$

Since $v_h|_K \in H^1(K)$, we may apply Green's theorem elementwise and obtain

$$\int_\Omega v_h\ D^\alpha z\ dx \ = \ \sum_{K\in\mathcal{T}_h} \int_K v_h\ D^\alpha z\ dx \ = \tag{3.23}$$
$$= \ - \sum_{K\in\mathcal{T}_h} \int_K D^\alpha v_h\ z\ dx \ + \ \sum_{K\in\mathcal{T}_h} \int_{\partial K} v_h\ n_\alpha D^\alpha z\ d\sigma \ =$$
$$= \ - \sum_{K\in\mathcal{T}_h} \int_K D^\alpha v_h\ z\ dx \ + \ \sum_{F\in\mathcal{F}_h(\Omega)} \int_F [v_h]\ n_\alpha D^\alpha z\ d\sigma\ ,$$

where $[v_h]$ denotes the jump $[v_h] := v_h|_{K_1} - v_h|_{K_2}$ across $F = K_1 \cap K_2, K_i \in \mathcal{T}_h, 1 \le i \le 2,$. But $v_h \in C^0(\overline{\Omega})$, and hence, $[v_h] = 0$ in (3.23) which proves (3.22) with $w_h^\alpha|_K := D^\alpha v_h|_K, K \in \mathcal{T}_h$. □

Corollary 3.1 H_0^1-conformity of finite elements

Let V_h be a finite element space and assume that

$$P_K \subset H^1(K) \quad , \quad K \in \mathcal{T}_h\ , \tag{3.24}$$

$$V_h \subset C_0^0(\Omega)\ . \tag{3.25}$$

Then

$$V_h \subset H_0^1(\Omega)\ . \tag{3.26}$$

Proof. The proof is left as an exercise. □

3.4 Lagrangian finite elements of type (k)

Definition 3.13 Polynomials of degree k

Let K be a d-simplex. For $k \ge 0$, we define $P_k(K)$ as the linear space of all polynomials of degree $\le k$ on K, i.e., $p \in P_k(K)$, if

$$p(x) \ = \ \sum_{|\alpha|\le k} a_\alpha\ x^\alpha \quad , \quad a_\alpha \in \mathbb{R}\ ,\ |\alpha| \le k\ ,$$

where

$$x^\alpha \ := \ \prod_{i=1}^d x_i^{\alpha_i} \quad , \quad \alpha \ := \ (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d \quad , \quad |\alpha| \ := \ \sum_{i=1}^d \alpha_i\ .$$

We note that

$$\dim\ P_k(K) \ = \ \binom{k+d}{d} \ = \ \frac{(k+d)!}{d!\ k!}\ . \tag{3.27}$$

Definition 3.14 Central lattice of order k of K

Let K be a d-simplex with vertices $a_i, 1 \le i \le d+1$, and $k \in \mathbb{N}_0$. The set

$$L_k(K) \ := \ \{ \ x \ = \ \sum_{i=1}^{d+1} \lambda_i \ a_i \ | \ \lambda_i \in \{0, \frac{1}{k}, ..., \frac{k-1}{k}, 1\} \ , \ \sum_{i=1}^{d+1} \lambda_i = 1 \ \}$$

is called the **central lattice of order** k of K. We have

$$\text{card } L_k(K) \ = \ \binom{k+d}{d} \ .$$

Definition 3.15 Nodal points

The elements of the central lattice of order k are called **nodal points**. For $D \subseteq \overline{\Omega}$, we refer to $\mathcal{N}_h(D)$ as the set of nodal points in D.

Definition 3.16 Lagrangian finite element of type k

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$. Then, the triple $(K, P_k(K), L_k(K))$, $K \in \mathcal{T}_h$ is called a **Lagrangian finite element of type** (k).

Lemma 3.4 Unisolvence of Lagrangian FEs of type (k)

A Lagrangian finite element of type (k) is **unisolvent**.

Proof. The proof is left as an exercise. □

Lemma 3.5 Affine equivalence of Lagrangian FEs of type (k)

Let $\mathbf{e}_i, 1 \le i \le d$, be the unit vectors in $\mathbb{R}^d$ with respect to the Cartesian coordinate system and

$$\hat{K} \ := \ \text{conv}(\mathbf{0}, \mathbf{e}_1, ..., \mathbf{e}_d) \ , \tag{3.28}$$

$$\hat{P}_{\hat{K}} \ := \ P_k(\hat{K}) \quad , \quad k \in \mathbb{N} \ ,$$

$$\hat{\Sigma}_{\hat{K}} \ := \ \{\hat{p}(\hat{x}) \ | \ \hat{x} \in L_k(\hat{K}) \ , \ \hat{p} \in \hat{P}_{\hat{K}}\} \ . \tag{3.29}$$

The Lagrangian finite elements of type k are **affine equivalent** to the **reference element** $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}}$. In particular, $\hat{K}$ is called the **reference d-simplex**.

Proof. Let $F_K : \mathbb{R}^d \to \mathbb{R}^d$ be the invertible affine mapping with $K = F_K(\hat{K})$. Then, $P_k(K) = \{p = \hat{p} \circ F_K^{-1} \ | \ \hat{p} \in P_k(\hat{K})\}$. Moreover, since $x \in L_k(K) \Longleftrightarrow x = F_K(\hat{x}), \hat{x} \in L_k(\hat{K})$, (3.17) is easily verified. □

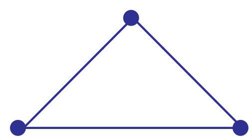

Fig. 3.3.a: $P_K = P_1(K)$ (d = 2)

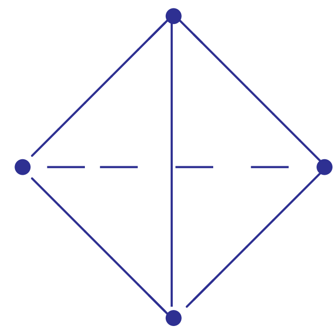

Fig. 3.3.b: $P_K = P_1(K)$ (d = 3)

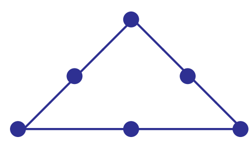

Fig. 3.4.a: $P_K = P_2(K)$ (d = 2)

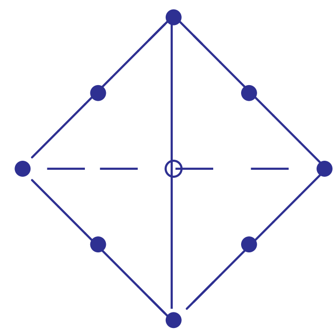

Fig. 3.4.b: $P_K = P_2(K)$ (d = 3)

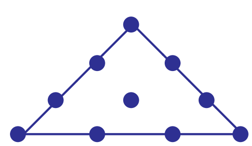

Fig. 3.5.a: $P_K = P_3(K)$ (d = 2)

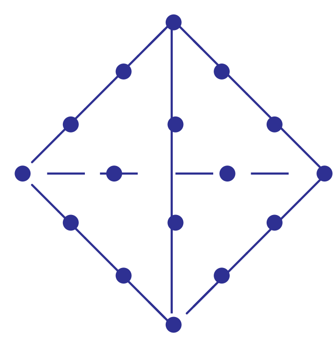

Fig. 3.5.b: $P_K = P_3(K)$ (d = 3)

Figures 3.3,3.4 and 3.5 illustrate Lagrangian finite elements of type (k) for $k = 1, 2, 3$ in two and in three dimensions.
Note that a Lagrangian finite element of type (1) for $d = 2$ is called **Courant's triangle** (cf. Fig. 3.3a).

Definition 3.17 Lagrangian finite element space

The finite element space V_h composed by Lagrangian finite elements of type (k) is called **Lagrangian finite element space** and denoted by $S_k(\Omega, \mathcal{T}_h)$.

In order to ensure **conformity** of $S_k(\Omega, \mathcal{T}_h)$, we have to require that the simplicial triangulation $\mathcal{T}_h$ is **geometrically conforming**.

Definition 3.18 Geometrically conforming simplicial triangulation

A simplicial triangulation $\mathcal{T}_h$ of the computational domain $\Omega \subset \mathbb{R}^d$ is called **geometrically conforming**, if there holds

($\mathcal{T}_h$5) The intersection of two different elements of the triangulation is either empty, or consists of a common face, or a common edge, or a common vertex.

Lemma 3.6 Conformity of Lagrangian finite element spaces

Let $\mathcal{T}_h$ be a geometrically conforming simplicial triangulation of the computational domain $\Omega \subset \mathbb{R}^d$. Then, there holds

$$S_k(\Omega, \mathcal{T}_h) = \{v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in P_k(K), K \in \mathcal{T}_h\} \subset H^1(\Omega). \tag{3.30}$$

Proof. Since $P_k(K) \subset H^1(K)$, $K \in \mathcal{T}_h$, in view of Theorem 3.1 we have only to show that $S_k(\Omega, \mathcal{T}_h) \subset C^0(\Omega)$. We give the proof exemplarily in case $d = 2$ and $k = 2$:
Let $K_i \in \mathcal{T}_h, 1 \leq i \leq 2,$ be two adjacent elements such that $E = K_1 \cap K_2 \in \mathcal{E}_h(\Omega)$ with nodal points $a_j \in \mathcal{N}_h(E), 1 \leq j \leq 3$. Let further $p_i \in P_2(K_i), 1 \leq i \leq 2$. Then, $p_i|_E \in P_2(E)$. Since $p_1(a_j) = p_2(a_j), 1 \leq j \leq 3,$ we conclude that $p_1|_E \equiv p_2|_E$. □

Corollary 3.2 Conformity of Lagrangian finite element spaces

Let $S_{k,0}(\Omega, \mathcal{T}_h) := \{v_h \in S_k(\Omega, \mathcal{T}_h) \mid v_h|_{\partial K \cap \partial\Omega} = 0, K \in \mathcal{T}_h, K \cap \partial\Omega \neq \emptyset\}$.
Then, under the same assumptions as in Lemma 3.5 there holds

$$S_{k,0}(\Omega, \mathcal{T}_h) \subset H_0^1(\Omega). \tag{3.31}$$

Proof. The proof is left as an exercise. □

Definition 3.19 Nodal basis functions
Let $\mathcal{N}_h(\overline{\Omega}) = \{x_j \mid 1 \le j \le n_h\}$. The function $\varphi_i \in S_k(\Omega, \mathcal{T}_h)$ given by

$$\varphi_i(x_j) = \delta_{ij} \quad , \quad 1 \le i, j \le n_h \ , \tag{3.32}$$

is called **nodal basis function with supporting point** $x_i \in \mathcal{N}_h(\overline{\Omega})$. The set of these functions constitutes a basis of $S_k(\Omega, \mathcal{T}_h)$.

Lemma 3.7 Representation by barycentric coordinates
Let K be a d-simplex with vertices $a_i, 1 \le i \le d+1$ and set

$$\begin{aligned} a_{ij} &:= \frac{1}{2}\,(a_i + a_j) \quad , \quad 1 \le i < j \le d+1 \ , \\ a_{iij} &:= \frac{1}{3}\,(2a_i + a_j) \quad , \quad 1 \le i \ne j \le d+1 \ , \\ a_{ijk} &:= \frac{1}{3}\,(a_i + a_j + a_k \quad , \quad 1 \le i < j < k \le d+1 \ . \end{aligned} \tag{3.33}$$

Let $\lambda_i, 1 \le i \le d$, be the **barycentric coordinates**. Then there holds:

(i) For $k = 1$, the nodal basis functions φ_i associated with the nodal points $a_i, 1 \le i \le d+1$, admit the representation

$$\varphi_i = \lambda_i \quad , \quad 1 \le i \le d+1 \ . \tag{3.34}$$

(ii) For $k = 2$, the nodal basis functions

- φ_i associated with the nodal points $a_i, 1 \le i \le d+1$, admit the representation

$$\varphi_i = \lambda_i\,(2\lambda_i - 1) \quad , \quad 1 \le i \le d+1 \ . \tag{3.35}$$

- φ_{ij} associated with the nodal points $a_{ij}, 1 \le i < j \le d+1$, admit the representation

$$\varphi_{ij} = 4\,\lambda_i\,\lambda_j \quad , \quad 1 \le i < j \le d+1 \ . \tag{3.36}$$

(iii) For $k = 3$, the nodal basis functions

- φ_i associated with the nodal points $a_i, 1 \le i \le d+1$, admit the representation

$$\varphi_i = \frac{1}{2}\,\lambda_i\,(3\lambda_i - 1)\,(3\lambda_i - 2) \quad , \quad 1 \le i \le d+1 \ . \tag{3.37}$$

- φ_{iij} associated with the nodal points $a_{iij}, 1 \le i \ne j \le d+1$, admit the representation

$$\varphi_{iij} = \frac{1}{2}\,9\,\lambda_i\,\lambda_j\,(3\lambda_i - 1) \quad , \quad 1 \le i \le d+1 \ . \tag{3.38}$$

- φ_{ijk} associated with the nodal points $a_{ijk}, 1 \leq i < j < k \leq d+1$, admit the representation

$$\varphi_{ijk} = 27\, \lambda_i\, \lambda_j\, \lambda_k \quad , \quad 1 \leq i < j < k \leq d+1 \; . \tag{3.39}$$

Proof. The proof is left as an exercise. □

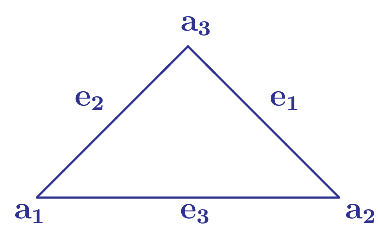

Fig. 3.6: Non degenerate triangle with vertices a_i and edges e_i

Lemma 3.8 Geometric characterization

Let K be a non degenerate 2-simplex with vertices $a_i, 1 \leq i \leq 3$, and edges $e_i, 1 \leq i \leq 3$ (cf. Fig. 3.6). Moreover, let $\mathbf{m}_{e_i}$ be the midpoints of the edges and denote by $\mathbf{n}_{e_i}$ the exterior unit normals to e_i. Then, the nodal basis functions $\varphi_i, 1 \leq i \leq 3$, associated with the vertices a_i admit the representation

$$\varphi_i(\mathbf{x}) = \frac{\text{meas}(e_i)}{2\text{meas}(K)}\, \mathbf{n}_{e_i} \cdot (\mathbf{m}_{e_i} - \mathbf{x}) \quad , \quad \mathbf{x} = (x_1, x_2)^T \; . \tag{3.40}$$

Proof. The proof is left as an exercise. □

3.5 Hermitian finite elements of type (3)

The name 'Lagrangian finite elements' is motivated by polynomial interpolation of Lagrangian type. Indeed, if $\Pi_K, K \in \mathcal{T}_h$, is the associated local interpolation operator, $\Pi_K v, v \in C^0(K)$, interpolates the function p in $x \in L_k(K)$.
Another type of polynomial interpolation is **Hermite interpolation**, where both point values and derivatives of a function are interpolated. The counterpart in finite elements are **Hermitian finite elements**. As an example, we consider Hermitian finite elements of type 3.

Definition 3.20 Hermitian finite elements of type (3)

Let K be a non degenerate d-simplex with vertices $a_i, 1 \leq i \leq d+1$, and $a_{ijk} := \frac{1}{3}(a_i + a_j + a_k), 1 \leq i < j < k \leq d+1$. Moreover, let

$$P_K := P_3(K) \; , \tag{3.41}$$

$$\begin{aligned}(3.42)\qquad \Sigma_K \ := \ \{ & p(a_i)\ ,\ 1 \le i \le d+1\ ,\\ & p(a_{ijk})\ ,\ 1 \le i < j < k \le d+1\ ,\\ & \frac{\partial p}{\partial x_j}(a_i)\ ,\ 1 \le i \le d+1\ ,\ 1 \le j \le d\}\ .\end{aligned}$$

Then, (K, P_K, Σ_K) is called a **Hermitian finite element of type** (3).

Remark 3.1 Equivalent specification of the DOFs

We note that in the definition of the degrees of freedom of the Hermitian finite element of type (3) the partial derivatives $\frac{\partial p}{\partial x_j}(a_i), 1 \le i \le d+1, 1 \le j \le d$, can be replaced by the **directional derivatives** $Dp(a_i)(a_j - a_i), 1 \le i \le d+1, 1 \le j \le d$, i.e., instead of Σ_K we have

$$\begin{aligned}(3.43)\quad \Sigma'_K \ := \ \{ & p(a_i)\ ,\ 1 \le i \le d+1\ ,\\ & p(a_{ijk})\ ,\ 1 \le i < j < k \le d+1\ ,\\ & Dp(a_i)(a_j - a_i\ ,\ 1 \le i \le d+1\ ,\ 1 \le j \le d\}\ .\end{aligned}$$

Lemma 3.9 Unisolvence of Hermitian finite elements

A Hermitian finite element of type (3) is **unisolvent**.

Proof. We give the proof exemplarily in case $d = 3$. Let $p \in P_3(K)$. It suffices to show that

$$(3.44)\qquad p(a_i) \ = \ 0\ ,\ 1 \le i \le 4\ ,$$

$$(3.45)\qquad p(a_{ijk}) \ = \ 0\ ,\ 1 \le i < j < k \le 4\ ,$$

$$(3.46)\qquad Dp(a_i)(a_j - a_i) \ = \ 0\ ,\ 1 \le i \le 4\ ,\ 1 \le j \le 3$$

implies $p \equiv 0$.

Let $F_{ijk}, 1 \le i < j < k \le 4$, be the face with vertices a_i, a_j, a_k and let e_{ij}, e_{ik}, e_{jk} be the corresponding edges. Then

$$p|_{e_{ij}} \in P_3(e_{ij}) \quad , \quad p|_{e_{ik}} \in P_3(e_{ik}) \quad , \quad p|_{e_{jk}} \in P_3(e_{jk})\ .$$

Since the **Hermite interpolation polynomial** interpolating

$$p(a_i)\ ,\ p(a_j)\ ,\ Dp(a_i)(a_j - a_i)\ ,\ Dp(a_j)(a_i - a_j)$$

is uniquely determined, (3.44) and (3.46) imply $p|_{e_{ij}} \equiv 0$. Likewise, we deduce $p|_{e_{ik}} \equiv 0$ and $p|_{e_{jk}} \equiv 0$, and hence

$$p|_{\partial F_{ijk}} \ \equiv \ 0\ .$$

Consequently, p is of the form

$$p \ = \ \alpha\ \lambda_i\ \lambda_j\ \lambda_k \quad , \quad \alpha \in \mathbb{R}\ .$$

Then, (3.45) implies $\alpha = 0$, i.e.,

$$p|_{F_{ijk}} \equiv 0 \; , \; 1 \le i < j < k \le 4 \quad \Longrightarrow \quad p \equiv 0 \; . \qquad \square$$

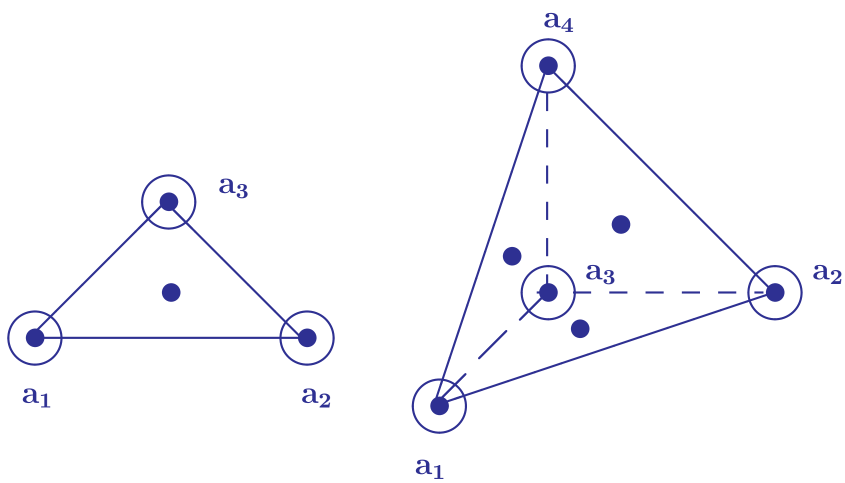

Figure 3.6: Hermitian finite element of type (3)

Figure 3.6 contains a schematic representation of Hermitian finite elements of type (3). The degrees of freedom, as given by point values, are marked by a dot, whereas the degrees of freedom, associated with partial resp. directional derivatives, are indicated by a circle.

Lemma 3.10 Affine equivalence of Hermitian finite elements
Let $\hat{K}$ be the **reference d-simplex** and

$$\hat{P}_{\hat{K}} \; := \; P_3(\hat{K}) \; ,$$

$$\begin{aligned} \hat{\Sigma}_{\hat{K}} \; := \; & \{\hat{p}(\hat{a}_i) \; , \; 1 \le i \le d+1 \; , \\ & \hat{p}(\hat{a}_{ijk}) \; , \; 1 \le i < j < k \le d+1 \; , \\ & \frac{\partial \hat{p}}{\partial \hat{x}_j}(\hat{a}_i) \; , \; 1 \le i \le d+1 \; , \; 1 \le j \le d\} \; . \end{aligned}$$

The Hermitian finite element $K, P_K, \Sigma_K)$ of type (3) is **affine equivalent** to the reference element $\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$ of type (3).

Proof. We only have to show how the partial derivatives are transformed. Let $F_K : \mathbb{R}^d \to \mathbb{R}^d$ be the invertible affine mapping $F_K(\hat{x}) = B_K\hat{x} + b_K$ such that $K = F_K(\hat{K})$. Then, for $\hat{p} \in P_3(\hat{K})$ and $p = \hat{p} \circ F_K^{-1}$

we have

$$\frac{\partial p}{\partial x_i}(a_j) \;=\; \sum_{k=1}^{d} \frac{\partial \hat{p}}{\partial \hat{x}_k} \, \frac{\partial (F_K^{-1}(a_j))_k}{\partial x_i} \ ,$$

whence

$$Dp(a_j) \;=\; B_K^{-1} \hat{D}\hat{p}(\hat{a}_j) \ . \quad \square$$

Definition 3.21 Hermitian finite element space

For a simplicial triangulation $\mathcal{T}_h$ of the polyhedral domain $\Omega \subset \mathbb{R}^d$, the finite element space V_h composed by Hermitian finite elements of type (3) is called a **Hermitian finite element space**. It will be denoted by $H_3(\Omega; \mathcal{T}_h)$.

Lemma 3.11 Conformity of Hermitian finite element spaces

Let $\mathcal{T}_h$ be a geometrically conforming simplicial triangulation of $\Omega \subset \mathbb{R}^d$. Then, the the Hermitian finite element space $H_3(\Omega; \mathcal{T}_h)$ is **conforming**, i.e.,

$$H_3(\Omega; \mathcal{T}_h) \subset H^1(\Omega) \ . \tag{3.47}$$

Proof. The proof is left as an exercise. $\square$

Definition 3.22 Basis functions of Hermitian finite element spaces

We restrict ourselves to the case $d = 2$ and assume that $H_3(\Omega; \mathcal{T}_h)$ is the Hermitian finite element space with respect to a geometrically conforming simplicial triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$.
We denote by $b_\ell, 1 \leq \ell \leq r_h$, the nodal points that are vertices of triangles $T \in \mathcal{T}_h$ and by $b_\ell, r_h + 1 \leq \ell \leq s_h$, those that are centers of gravity of $T \in \mathcal{T}_h$. Then, an appropriate basis

$$\varphi_k \ , \ 1 \leq k \leq s_h \quad , \quad \varphi_k^{(\nu)} \ , \ 1 \leq k \leq r_h \ , \ 1 \leq \nu \leq 2 \ , \tag{3.48}$$

is given by

$$\begin{aligned} \varphi_k(b_\ell) \;&=\; \delta_{k\ell} \ , \ 1 \leq k,\ell \leq s_h \ , \\ \frac{\partial \varphi_k}{\partial x_\nu}(b_\ell) \;&=\; 0 \ , \ 1 \leq k \leq s_h \ , \ 1 \leq \ell \leq r_h \ , \ 1 \leq \nu \leq 2 \ , \end{aligned} \tag{3.49}$$

$$\begin{aligned} \varphi_k^{(1)}(b_\ell) \;&=\; 0 \ , \ 1 \leq k \leq r_h \ , \ 1 \leq \ell \leq s_h \ , \\ \frac{\partial \varphi_k^{(1)}}{\partial x_1}(b_\ell) \;&=\; \delta_{k\ell} \ , \ \frac{\partial \varphi_k^{(1)}}{\partial x_2}(b_\ell) \;=\; 0 \ , \ 1 \leq k,\ell \leq r_h \ , \end{aligned} \tag{3.50}$$

$$(3.51) \qquad \varphi_k^{(2)}(b_\ell) = 0 \ , \ 1 \le k \le r_h \ , \ 1 \le \ell \le s_h \ ,$$
$$\frac{\partial \varphi_k^{(2)}}{\partial x_1}(b_\ell) = 0 \ , \ \frac{\partial \varphi_k^{(2)}}{\partial x_2}(b_\ell) = \delta_{k\ell} \ , \ 1 \le k, \ell \le r_h \ .$$

3.6 Lagrangian finite elements of type $[k]$

Let $K := \prod_{i=1}^{d} [c_i, d_i] \subset \mathbb{R}^d$ be a d-rectangle. For $k \in \mathbb{N}_0$ we denote by $Q_k(K)$ the linear space of all polynomials that are of degree $\le k$ in each of the d variables $x_i, 1 \le i \le d$, i.e., $p \in Q_k(K)$ is of the form

$$(3.52) \qquad p(x) = \sum_{\alpha_i \le k} a_{\alpha_1 \ldots \alpha_d} \ x_1^{\alpha_1} \ x_2^{\alpha_2} \ldots x_d^{\alpha_d} \ ,$$

from which we easily deduce

$$(3.53) \qquad \dim \ Q_k(K) = (k+1)^d \ .$$

For the d-rectangle K we consider the point-set

$$(3.54) \qquad L_{[k]}(K) := \{ x = (x_{i_1}, \ldots, x_{i_d})^T \mid x_{i_\ell} := c_\ell + \frac{i_\ell}{k}(d_\ell - c_\ell) \ , \ i_\ell \in \{0, 1, \ldots, k\} \ , \ 1 \le \ell \le d \}$$

and define

$$(3.55) \qquad \Sigma_K := \{ p(x) \mid x \in L_{[k]}(K) \} \ , \ p \in Q_k(K) \ .$$

Then there holds:

Definition 3.22 Lagrangian finite element of type $([k])$
The element $(K, Q_k(K), \Sigma_K)$ is called a **Lagrangian finite element of type** $[k]$ and will be denoted by $S_{[k]}(K)$. The points $x \in L_{[k]}(K)$ are refereed to as **nodal points**.

Lemma 3.12 Lagrangian finite element of type $([k])$ as a tensor product finite element
The Lagrangian finite element of type $([k])$ is a tensor product finite element based on **tensor product Lagrangian type polynomial interpolation**:
The polynomial $p \in Q_k(K)$, interpolating in $x \in L_{[k]}(K)$, has the representation

$$(3.56) \qquad p(x) = \sum_{y \in L_{[k]}(K)} L(x) \ p(y) \ .$$

Here, the **Lagrangian fundamental polynomial** $L \in Q_k(K)$ is given by

$$L(x) \;=\; \prod_{\ell=1}^{d} L_{i_\ell}(x) \tag{3.57}$$

in terms of the **one-dimensional Lagrangian fundamental polynomials**

$$L_{i_\ell}(x) \;:=\; \prod_{i_\ell \neq i'_\ell = 0}^{k} \frac{x - x_{i'_\ell}}{x_{i_\ell} - x_{i'_\ell}} \quad , \quad 1 \leq \ell \leq d \; . \tag{3.58}$$

Proof. Let $p \in Q_k(K)$ and consider an arbitrary edge $e_j \subset K, j \in \{1, d\}$ as given by

$$e_j \;:=\; [c_j, d_j] \times \prod_{j \neq \ell = 1}^{d} \{g_\ell\} \quad , \quad g_\ell \in \{c_\ell, d_\ell\} \; .$$

Then, $p|_{e_j} \in P_k(e_j)$ is uniquely determined by its values in $x_{i_j} := c_j + \frac{i_j}{k}(d_j - c_j) \in e_j, i_j \in \{0, 1, ..., k\}$, and has the Lagrangian representation

$$p|_{e_j}(x) \;=\; \sum_{i_j=0}^{k} L_{i_j}(x) \; p|_{e_j}(x_{i_j}) \quad , \quad x \in e_j \; . \quad \square$$

Lemma 3.13 Unisolvence

The Lagrangian finite element of type $([k])$ is **unisolvent**.

Proof. The proof is an immediate consequence of Lemma 3.12. $\square$

Lemma 3.14 Affine equivalence of Lagrangian FEs of type $[k]$

Let $\hat{K} := [0, 1]^d$ be the **unit cube** in $\mathbb{R}^d$ and

$$\hat{P}_{\hat{K}} \;:=\; Q_k(\hat{K}) \quad , \quad k \in \mathbb{N} \; , \tag{3.59}$$

$$\hat{\Sigma}_{\hat{K}} \;:=\; \{\hat{p}(\hat{x}) \mid \hat{x} \in L_{[k]}(\hat{K}) \; , \; \hat{p} \in \hat{P}_{\hat{K}}\} \; . \tag{3.60}$$

The Lagrangian finite elements of type $[k]$ are **affine equivalent** to the **reference element** $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}}$. In particular, $\hat{K}$ is called the **reference d-rectangle**.

Proof. The proof is left as an exercise. $\square$

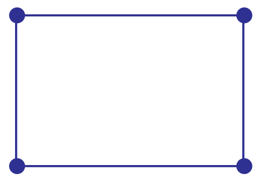

Fig. 3.7.a: $S_{[1]}(K)$ (d = 2)

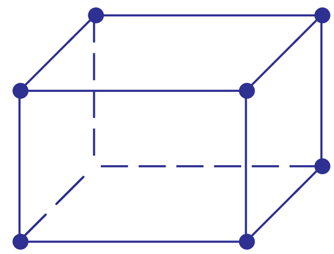

Fig. 3.7.b: $S_{[1]}(K)$ (d = 3)

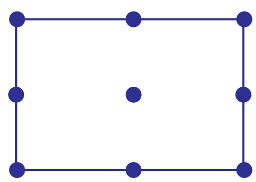

Fig. 3.8.a: $S_{[2]}(K)$ (d = 2)

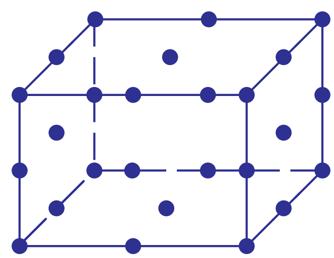

Fig. 3.8.b: $S_{[2]}(K)$ (d = 3)

For $k = 1$ and $k = 2$, Figures 3.7 and 3.8 contain illustrations of Lagrangian finite elements of type $[k]$.

Definition 3.23 Lagrangian finite element space
Let $\Omega \subset \mathbb{R}^d$ be given as the union of a finite number of d-rectangles and let $\mathcal{T}_h$ be a rectangular triangulation of Ω. The finite element space V_h composed by Lagrangian finite elements of type $[k]$ is called **Lagrangian finite element space** and denoted by $S_{[k]}(\Omega, \mathcal{T}_h)$.

Lemma 3.15 Conformity of Lagrangian finite element spaces
Let $\mathcal{T}_h$ be a geometrically conforming rectangular triangulation of the computational domain Ω. Then, there holds

$$S_{[k]}(\Omega, \mathcal{T}_h) = \{v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in Q_k(K), K \in \mathcal{T}_h\} \subset H^1(\Omega). \tag{3.61}$$

Proof. The proof is left as an exercise □

Corollary 3.3 Conformity of Lagrangian finite element spaces
Let $S_{[k],0}(\Omega,\mathcal{T}_h) := \{v_h \in S_{[k]}(\Omega,\mathcal{T}_h) \mid v_h|_{\partial K\cap\partial\Omega} = 0, K \in \mathcal{T}_h, K \cap \partial\Omega \neq \emptyset\}$. Then, under the same assumptions as in Lemma 3.15 there holds

$$S_{[k],0}(\Omega,\mathcal{T}_h) \subset H_0^1(\Omega). \tag{3.62}$$

Proof. The proof is left as an exercise. □

Definition 3.24 Nodal basis functions
Let $\mathcal{N}_h(\overline{\Omega}) = \{x \in L_{[k]}(K) \mid K \in \mathcal{T}_h\}$ and suppose $\mathrm{card}(\mathcal{N}_h(\overline{\Omega})) = n_h$. The function $\varphi_i \in S_{[k]}(\Omega,\mathcal{T}_h)$ given by

$$\varphi_i(x_j) \;=\; \delta_{ij} \quad , \quad 1 \leq i,j \leq n_h \ , \tag{3.63}$$

is called **nodal basis function with supporting point** $x_i \in \mathcal{N}_h(\overline{\Omega})$. The set of these functions constitutes a basis of $S_{[k]}(\Omega,\mathcal{T}_h)$.