# Chapter 4 A priori error estimates for conforming finite element approximations

## 4.1 Interpolation in Sobolev spaces

For a simply-connected Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $m \in \mathbb{N}_0$ we consider the **quotient space** $W^{m+1,p}(\Omega)/P_m(\Omega), p \in [1,\infty],$ whose elements are **equivalence classes** $[u]$ according to

$$[u] \ := \ \{ \ w \in W^{m+1,p}(\Omega) \mid w - u \in P_m(\Omega) \ \} \ .$$

We recall that $W^{m+1,p}(\Omega)/P_m(\Omega)$ is a Banach space with respect to the **Sobolev quotient norm**

(4.1) $$\|[u]\|_{m+1,p,\Omega} \ := \ \inf_{p \in P_m(\Omega)} \|u + p\|_{m+1,p,\Omega} \ .$$

A basic result states the equivalence of the quotient norm with the $|\cdot|_{m+1,p,\Omega}$-semi-norm.

**Theorem 4.1 Equivalence of the Sobolev quotient norm and the Sobolev semi-norm**

Let $\Omega \subset \mathbb{R}^d$ be a simply-connected Lipschitz domain and $m \in \mathbb{N}_0$. Then, there exists a constant $C(\Omega) \geq 0$ such that for all

$$[u] \in W^{m+1,p}(\Omega)/P_m(\Omega) \quad , \quad p \in [1,\infty] \ ,$$

there holds

(4.2) $$|u|_{m+1,p,\Omega} \leq \ \|[u]\|_{m+1,p,\Omega} \ \leq \ C(\Omega) \ |u|_{m+1,p,\Omega} \ .$$

**Proof.** The first inequality in (4.2) is trivial. For the proof of the second one, let $N_m := \dim \ P_m(\Omega)$ and denote by $\ell_i, 1 \leq i \leq N_m$, a basis of the dual space of $P_m(\Omega)$. In particular, we have

(4.3) $$\ell_i(p) = 0 \ , \ 1 \leq i \leq N_m \ , \ p \in P_m(\Omega) \Longleftrightarrow p = 0 \ .$$

By the **Hahn-Banach extension theorem** there exist functionals $\tilde{\ell}_i \in W^{m+1,p}(\Omega)^*, 1 \leq i \leq N_m$, such that $\tilde{\ell}_i|_{P_m(\Omega)} = \ell_i$. We will show

(4.4) $$\|u\|_{m+1,p,\Omega} \ \leq \ C(\Omega) \left( |u|_{m+1,p,\Omega} + \sum_{i=1}^{N_m} |\tilde{\ell}_i(u)| \right) \ , \ u \in W^{m+1,p}(\Omega) \ .$$

by a **contradiction argument**. If (4.4) does not hold true, there exists a sequence $(u_k)_{\mathbb{N}}, u_k \in W^{m+1,p}(\Omega)$, such that

(4.5)

$$\|u_k\|_{m+1,p,\Omega} = 1 \ , \ k \in \mathbb{N} \quad , \quad \lim_{k \to \infty} \left( |u_k|_{m+1,p,\Omega} \ + \ \sum_{i=1}^{N_m} |\tilde{\ell}_i(u_k)| \right) = 0 \ .$$

Since the sequence $(u_k)_{\mathbb{N}}$ is bounded in $W^{m+1,p}(\Omega)$ and $W^{m+1,p}(\Omega)$ is compactly embedded in $W^{m,p}(\Omega)$, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and $u \in W^{m,p}(\Omega)$ such that

$$\|u_k - u\|_{m,p,\Omega} \to 0 \ (k \in \mathbb{N}' \ , \ k \to \infty) \ . \tag{4.6}$$

On the other hand, it follows readily from (4.5) that

$$\|u_k\|_{m+1,p,\Omega} \to 0 \ (k \in \mathbb{N}' \ , \ k \to \infty) \ . \tag{4.7}$$

Therefore, in view of (4.6) we must have that $u_k \to u$ $(k \in \mathbb{N}', k \to \infty)$ in $W^{m+1,p}(\Omega)$ with

$$\|D^\alpha u\|_{0,\Omega} \ = \ \lim_{k\to\infty} \|D^\alpha u_k\|_{0,\Omega} \ = \ 0 \quad , \quad |\alpha| = m+1 \ .$$

Hence, $D^\alpha u = 0, |\alpha| = m+1$ which in view of the connectedness of $\Omega$ implies $u \in P_m(\Omega)$. Then (4.5) shows

$$\tilde{\ell}_i(u) \ = \ \lim_{k\to\infty} \tilde{\ell}_i(u_k) \ = \ 0 \quad , \quad 1 \le i \le N_m \ .$$

Since $\tilde{\ell}_i(u) = \ell_i(u)$, (4.3) tells us that $u = 0$ which contradicts (4.5). □

As a consequence of the previous theorem we obtain the celebrated **Bramble-Hilbert lemma**.

**Theorem 4.2 Bramble-Hilbert lemma**

Under the assumptions of Theorem 4.1, let $\ell \in W^{(m+1),p}(\Omega)^*, p \in [1,\infty]$, such that $P_m(\Omega) \subset \text{Ker } \ell$. Then, there exists a constant $C(\Omega) > 0$ such that for all $u \in W^{m+1,p}(\Omega)$

$$|\ell(u)| \ \le \ C(\Omega) \ \|\ell\|^*_{m+1,p,\Omega} |u|_{m+1,\Omega} \ . \tag{4.8}$$

**Proof.** For $p \in P_m(\Omega)$ we have

$$|\ell(u)| \ = \ |\ell(u+p)| \ \le \ \|\ell\|^*_{m+1,p,\Omega} \ \|u+p\|_{m+1,p,\Omega} \ . \quad \square$$

Based on the Bramble-Hilbert lemma, we will estimate the interpolation error $u - \Pi_K u, K \in \mathcal{T}_h(\Omega)$, where $\Pi_K$ stands for the **local interpolation operator**. Since we do not want to invoke estimates with a constant that depends on the actual elements of the triangulation, we will make essential use of the **affine equivalence** of finite elements. Therefore, we have to know how Sobolev norms and semi-norms transform under an affine mapping.

We assume that $\Omega \subset \mathbb{R}^d$ and $\hat{\Omega} \subset \mathbb{R}^d$ are **affine equivalent** open subsets, i.e., there exists a regular affine mapping

$$\begin{aligned} F \ : \ \hat{\Omega} \ &\to \ \Omega \\ \hat{x} \ &\to \ F(\hat{x}) = B\hat{x} + b \end{aligned} \tag{4.9}$$

such that $F(\hat{\Omega}) = \Omega$.

**Theorem 4.3 Sobolev norms under affine mappings**

Let $\Omega \subset \mathbb{R}^d$ and $\hat{\Omega} \subset \mathbb{R}^d$ be **affine equivalent** Lipschitz domains and assume $v \in W^{m,p}(\Omega), m \in \mathbb{N}_0, p \in [1,\infty]$. Then, for the function $\hat{v} := v \circ F$ there holds $\hat{v} \in W^{m,p}(\hat{\Omega})$, and there exists a positive constant $C = C(m,d)$ such that

$$(4.10) \quad |\hat{v}|_{m,p,\hat{\Omega}} \;\leq\; C\, \|B\|^m\, |\det(B)|^{-1/p}\, |v|_{m,p,\Omega} \quad , \quad v \in W^{m,p}(\Omega) \ ,$$

and

$$(4.11) \quad |v|_{m,p,\Omega} \;\leq\; C\, \|B^{-1}\|^m\, |\det(B)|^{1/p}\, |\hat{v}|_{m,p,\hat{\Omega}} \quad , \quad \hat{v} \in W^{m,p}(\hat{\Omega}) \ .$$

**Proof.** For the proof of (4.10) in case $p \in [1,\infty)$ we first assume $v \in C^m(\overline{\Omega})$. Then, $\hat{v} \in C^m(\hat{\Omega})$ and for any multiindex $\alpha$ with $|\alpha| = m$, we obtain

$$D^\alpha \hat{v}(\hat{x}) \;=\; D^m \hat{v}(\hat{x})(e_{\alpha_1}, ..., e_{\alpha_m}) \ .$$

Here, $D^m \hat{v}(\hat{x}) \in \mathcal{L}(\mathbb{R}^{md})$ is the $m$-th linear total differential of $\hat{v}$ and $e_{\alpha_i} \in \{e_1, ..., e_d\}, 1 \leq i \leq m$, where $\{e_1, ..., e_d\}$ denotes the Cartesian basis of $\mathbb{R}^d$. It follows that

$$|D^\alpha \hat{v}(\hat{x})| \;\leq\; \|D^m \hat{v}(\hat{x})\| \;=\; \sup_{\|\xi_i\| \leq 1} |D^m \hat{v}(\hat{x})(\xi_1, ..., \xi_m)| \ ,$$

and hence,

$$(4.12) \qquad |\hat{v}|_{m,p,\hat{\Omega}} \;=\; \Big( \int_{\hat{\Omega}} \sum_{|\alpha|=m} |D^\alpha \hat{v}(\hat{x})|^p \, d\hat{x} \Big)^{1/p} \;\leq$$

$$\leq\; \underbrace{(\mathrm{card}(\{\alpha \in \mathbb{N}_0^d \mid |\alpha| = m\})^{1/p}}_{=: \ C_1(m,d)} \Big( \int_{\hat{\Omega}} |D^m \hat{v}(\hat{x})|^p \, d\hat{x} \Big)^{1/p} \ .$$

Now, by the chain rule

$$D^m \hat{v}(\hat{x})(\xi_1, ..., \xi_m) \;=\; D^m v(x)(B\xi_1, ..., B\xi_m) \ ,$$

whence

$$\|D^m \hat{v}(\hat{x})\| \;\leq\; \|D^m v(x)\| \; \|B\|^m \ ,$$

and thus

$$(4.13) \qquad \int_{\hat{\Omega}} \|D^m \hat{v}(\hat{x})\|^p \, d\hat{x} \;\leq\; \|B\|^m \int_{\hat{\Omega}} \|D^m (v(F(\hat{x}))\|^p \, d\hat{x} \ .$$

Applying the rule for change of variables in multiple integrals, we obtain

$$(4.14) \qquad \int_{\hat{\Omega}} \|D^m (v(F(\hat{x}))\|^p \, d\hat{x} \;=\; |\det(B^{-1})| \int_{\Omega} \|D^m v(x)\|^p \, dx \ .$$

Moreover, there exists a positive constant $C_2(m,d)$ such that

$$\|D^m v(x)\| \ = \ \sup_{\|\xi_i\|\le 1} D^m v(x)(\xi_1,...,\xi_m) \ \le \ C_2(m,d) \ \max_{|\alpha|=m} |D^\alpha v(x)| \ ,$$

whence

$$(4.15) \qquad (\int_\Omega \|D^m v(x)\|^p \ dx)^{1/p} \ \le \ C_2(m,d) \ |v|_{m,p,\Omega} \ .$$

For $v \in C^m(\overline{\Omega})$, (4.10) follows readily from (4.12)-(4.15).
Finally, using the continuity of the operator

$$\begin{aligned} (C^m(\overline{\Omega}), \|\cdot\|_{m,p,\Omega}) \quad &\to \quad W^{m,p}(\hat{\Omega}) \\ v \quad &\longmapsto \quad \hat{v} := v \circ F \end{aligned}$$

and the denseness of $C^m(\overline{\Omega})$ in $W^{m,p}(\Omega)$, proves (4.10) for $v \in W^{m,p}(\Omega)$.
The proof of (4.10) in case $p = \infty$ is left as an exercise.
The proof of (4.11) can be established in an analogous way. □

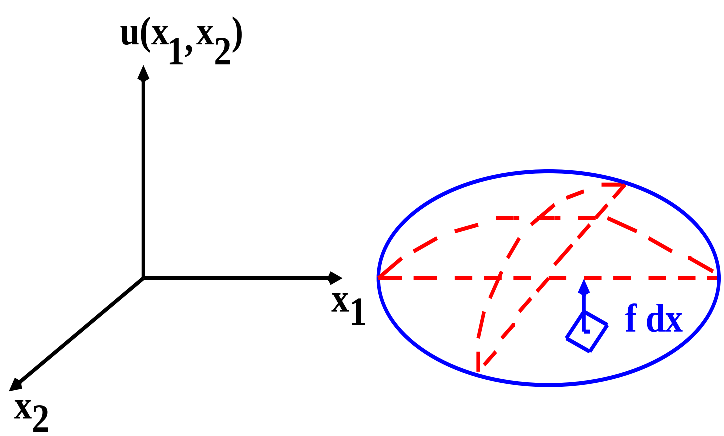


Figure 4.1: Affine equivalence of Lipschitz domains

The terms $\|B\|$ and $\|B^{-1}\|$ in (4.10) and (4.11) can be further estimated by geometric quantities associated with the affine equivalent Lipschitz domains. For that purpose, we set

$$(4.16) \qquad h \ := \ \text{diam}(\Omega) \quad , \quad \hat{h} \ := \ \text{diam}(\hat{\Omega})$$

and

$$(4.17) \qquad \begin{aligned} \rho \ &:= \ \sup \ \{\text{diam}(B) \mid B \text{ is a ball with } B \subset \Omega\} \ , \\ \hat{\rho} \ &:= \ \sup \ \{\text{diam}(\hat{B}) \mid \hat{B} \text{ is a ball with } \hat{B} \subset \hat{\Omega}\} \ . \end{aligned}$$

**Lemma 4.1 Geometry of affine equivalent Lipschitz domains**
Let $\hat{\Omega} \subset \mathbb{R}^d$ and $\Omega = F(\hat{\Omega}) \subset \mathbb{R}^d$ be two affine equivalent Lipschitz domains, where $F(\hat{x}) := B\hat{x} + b$ is an invertible affine mapping. Moreover, let $h, \hat{h}$ and $\rho, \hat{\rho}$ be given by (4.16) and (4.17). Then, there holds

$$\|B\| \le \frac{h}{\hat{\rho}} \quad , \quad \|B^{-1}\| \le \frac{\hat{h}}{\rho} \, , \tag{4.18}$$

$$|\det(B)| = \frac{\text{meas}(\Omega)}{\text{meas}(\hat{\Omega})} \, . \tag{4.19}$$

**Proof.** For the proof of the first inequality in (4.18) we start from

$$\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\hat{w}\| = \hat{\rho}} \|B\hat{w}\| \, .$$

For $\hat{w} \in \mathbb{R}^d$ with $\|\hat{w}\| = \hat{\rho}$ there exist $\hat{y}, \hat{z} \in \mathbb{R}^d$ with $\hat{w} = \hat{y} - \hat{z}$ (see Figure 4.1). Setting $y := F(\hat{y}), z := F(\hat{z})$, we have $w := y - z = B(\hat{y} - \hat{z}) = B(\hat{w})$, and consequently $\|B\hat{w}\| \le h$.
The proof of the second inequality in (4.18) follows along the same lines. The proof of (4.19) is left as an exercise. □

**Theorem 4.4 Properties of polynomial preserving operators**
Let $\hat{\Omega} \subset \mathbb{R}^d$ be a Lipschitz domain and assume that $k, m \in \mathbb{N}_0$ and $p, q \in [1, \infty]$ are such that $W^{k+1,p}(\hat{\Omega})$ is **continuously embedded** in $W^{m,q}(\hat{\Omega})$. Moreover, assume that

$$\hat{\Pi} \, : \, W^{k+1,p}(\hat{\Omega}) \to W^{m,q}(\hat{\Omega})$$

is a **polynomial preserving linear operator** in the sense that

$$\hat{\Pi}\hat{p} = \hat{p} \quad , \quad \hat{p} \in P_k(\hat{\Omega}) \, . \tag{4.20}$$

Let $\Omega \subset \mathbb{R}^d$ be **affine equivalent** to $\hat{\Omega}$ and let

$$\Pi_\Omega \, : \, W^{k+1,p}(\Omega) \to H^{m,q}(\Omega)$$

be defined by

$$\widehat{\Pi_\Omega v} = \hat{\Pi}\hat{v} \quad \text{for all } \hat{v} = v \circ F^{-1} \, , \; v \in W^{k+1,p}(\Omega) \, . \tag{4.21}$$

Then, there exists a positive constant $C(\hat{\Pi}, \hat{\Omega})$ such that for all $v \in W^{k+1,p}(\Omega)$

$$|v - \Pi_\Omega v|_{m,q,\Omega} \le C(\hat{\Pi}, \hat{\Omega}) \, (\text{meas}(\Omega))^{\frac{1}{q} - \frac{1}{p}} \, \frac{h^{k+1}}{\rho^m} \, |v|_{k+1,p,\Omega} \, . \tag{4.22}$$

**Proof.** Denoting by $\hat{I} : W^{k+1,p}(\hat{\Omega}) \to W^{m,q}(\hat{\Omega})$ the embedding operator, in view of the polynomial preserving property (4.21) we have

$$\hat{v} - \hat{\Pi}\hat{v} \;=\; (\hat{I} - \hat{\Pi})(\hat{v} + \hat{p}) \quad , \quad \hat{v} \in W^{k+1,p}(\hat{\Omega}) \ , \ \hat{p} \in P_k(\hat{\Omega}) \ .$$

Application of the **Bramble-Hilbert lemma** yields

(4.23)
$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \le \|\hat{I} - \hat{\Pi}\| \inf_{\hat{p} \in P_k(\hat{\Omega})} \|\hat{v} + \hat{p}\|_{k+1,p,\hat{\Omega}} \le C(\hat{\Pi}, \hat{\Omega}) \ \|\hat{v}|_{k+1,p,\hat{\Omega}} \ .$$

Moreover, due to (4.21) we have

$$\hat{v} - \hat{\Pi}\hat{v} \;=\; \widehat{v - \Pi_\Omega v} \ ,$$

and hence, using Theorem 4.3 results in

$$|v - \Pi_\Omega v|_{m,q,\Omega} \;\le\; C \ \|B^{-1}\|^m \ |\det(B)|^{1/q} \ |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \ . \tag{4.24}$$

Applying Theorem 4.3 once more, we get

$$|\hat{v}|_{k+1,p,\hat{\Omega}} \;\le\; C \ \|B\|^{k+1} \ |\det(B)|^{-1/p} \ |v|_{k+1,p,\Omega} \ . \tag{4.25}$$

The assertion follows from (4.23)-(4.25), (4.18),(4.19), and

$$|\det(B_K)| \;=\; \frac{\text{meas}(\Omega)}{\text{meas}(\hat{\Omega})} \ .$$

. □

## 4.2 Affine equivalent finite elements and shape regular triangulations

The estimates for the interpolation error derived in the previous section will now be applied to families of affine equivalent finite elements. If $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and $\mathcal{T}_h$ a geometrically conforming triangulation of $\Omega$, for $K \in \mathcal{T}_h$ we introduce

$$h_K \;:=\; \text{diam}(K) \ , \tag{4.26}$$
$$\rho_K \;:=\; \sup \ \{\text{diam}(B) \mid B \text{ is a ball with } B \subset K\} \ . \tag{4.27}$$

**Theorem 4.5 Estimation of the local interpolation error**

Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a finite element and let $s \in \mathbb{N}_0$ be the largest order of partial derivatives that occurs in the definition of the degrees of freedom $\hat{\Sigma}$. Assume further that the integers $k, m \in \mathbb{N}_0$ are such that the mappings

$$H^{k+1}(\hat{K}) \;\to\; C^s(\hat{K}) \ , \tag{4.28}$$
$$H^{k+1}(\hat{K}) \;\to\; H^m(\hat{K}) \tag{4.29}$$

are **continuous inclusions** and there holds

$$P_k(\hat{K}) \subset \hat{P} \subset H^m(\hat{K}) \ . \tag{4.30}$$

Then, there exists a positive constant $C(\hat{K},\hat{P},\hat{\Sigma})$ such that for all elements $(K,P_K,\Sigma_K)$ that are affine equivalent to $(\hat{K},\hat{P},\hat{\Sigma})$

$$|v - \Pi_K v|_{m,K} \ \le \ C(\hat{K},\hat{P},\hat{\Sigma}) \ \frac{h_K^{k+1}}{\rho_K^m} \ |v|_{k+1,K} \quad , \quad v \in H^{k+1}(K) \ , \tag{4.31}$$

where $\Pi_K : C^s(K) \to P_K$ stands for the local interpolation operator.

**Proof.** Without loss of generality we assume $s = 2$.
In view of (4.30) we have

$$\hat{\Pi}_{\hat{K}} \hat{p} \ = \ \hat{p} \quad , \quad \hat{p} \in P_k(\hat{K}) \ . \tag{4.32}$$

Moreover, if $\hat{v} \in H^{k+1}(\hat{K})$, then $\hat{v} \in C^s(\hat{K})$ due to (4.28). Hence, $\hat{\Pi}_{\hat{K}}\hat{v}$ is well defined and has the representation

$$\hat{\Pi}_{\hat{K}}\hat{v} = \sum_i \hat{v}(\hat{a}_i^0)\hat{p}_i^0 + \sum_{i,j} D\hat{v}(\hat{a}_l^1)(\hat{\xi}_{ij}^1)\hat{p}_{ij}^1 + \sum_{i,j,\ell} D^2\hat{v}(\hat{a}_i^2)(\hat{\xi}_{ij}^2\hat{\xi}_{i\ell}^2)\hat{p}_{ij\ell}^2 \ .$$

Observing (4.28),(4.29) and using this representation, the **continuity** of $\hat{\Pi}_{\hat{K}} : H^{k+1}(\hat{K}) \to H^m(\hat{K})$ can be deduced as follows

$$\begin{aligned}
\|\hat{\Pi}_{\hat{K}}\hat{v}\|_{m,\hat{K}} &\le \sum_i |\hat{v}(\hat{a}_i^0)| \ \|\hat{p}_i^0\|_{m,\hat{K}} + \sum_{i,j} |D\hat{v}(\hat{a}_l^1)(\hat{\xi}_{ij}^1)| \ \|\hat{p}_{ij}^1\|_{m,\hat{K}} + \\
&+ \sum_{i,j,\ell} |D^2\hat{v}(\hat{a}_i^2)(\hat{\xi}_{ij}^2\hat{\xi}_{i\ell}^2)| \ \|\hat{p}_{ij\ell}^2\|_{m,\hat{K}} \ \le \\
&\le C(\|\hat{p}_i^0\|_{m,\hat{K}}, \|\hat{\xi}_{ij}^1\|\|\hat{p}_{ij}^1\|_{m,\hat{K}}, \|\hat{\xi}_{ij}^2\|\|\hat{\xi}_{i\ell}^2\|\|\hat{p}_{ij\ell}^2\|_{m,\hat{K}}) \ \sup_{|\alpha|\le 2} \sup_{\hat{x}\in\hat{K}} |D^\alpha \hat{v}(\hat{x})| \\
&\le C(\hat{K},\hat{P},\hat{\Sigma}) \ \|\hat{v}\|_{k+1,\hat{K}} \ .
\end{aligned}$$

Since the local interpolation operators $\hat{\Pi}_{\hat{K}}$ and $\Pi_K$ are related according to

$$\widehat{\Pi_K v} \ = \ \hat{\Pi}_{\hat{K}}\hat{v} \quad , \quad v \in C^s(K) \ ,$$

the estimate (4.31) follows from Theorem 3.4. □

For so-called **shape regular** triangulations $\mathcal{T}_h$, the quantity $\rho_K$ can be eliminated from the previous estimate.

**Definition 4.1 Shape regular triangulations**

A triangulation $\mathcal{T}_h$ of a Lipschitz domain $\Omega \subset \mathbb{R}^d$ is called **shape regular**, if there exists a positive real number $\sigma$, independent of $h_K, K \in \mathcal{T}_h$, such that for all $K \in \mathcal{T}_h$

$$\frac{h_K}{\rho_K} \le \sigma . \tag{4.33}$$

**Corollary 4.1 Estimation of the local interpolation error**

Let $\mathcal{T}_h$ be a shape regular triangulation of a Lipschitz domain $\Omega \subset \mathbb{R}^d$. Assume that $(K, P_K, \Sigma_K), K \in \mathcal{T}_h$, are finite elements that are affine equivalent to the reference element $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$ which satisfies the assumptions of Theorem 4.5. Then, there exists a positive constant $C(\hat{K}, \hat{P}, \hat{\Sigma})$ such that for all $K \in \mathcal{T}_h$

$$|v - \Pi_K v|_{m,K} \le C(\hat{K}, \hat{P}, \hat{\Sigma}) h_K^{k+1-m} |v|_{k+1,K} \quad , \quad v \in H^{k+1}(K) . \tag{4.34}$$

## 4.3 A priori estimate in the $H^1(\Omega)$-norm

In this section, we will derive **a priori estimates** for the finite element discretization of **second order elliptic boundary value problems** by **finite elements** with respect to a triangulation $\mathcal{T}_h$ of the computational domain $\Omega \subset \mathbb{R}^d$ under the following **assumptions**:

**(A1)** The triangulation $\mathcal{T}_h$ is **shape regular**.

**(A2)** $(K, P_K, \Sigma_K)_{K \in \mathcal{T}_h}$ is a **family of affine equivalent finite elements** with reference element $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$.

**(A3)** The finite elements $(K, P_K, \Sigma_K), K \in \mathcal{T}_h$, are $C^0$**-elements**.

We consider the case where $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$ and $V_h \subset V$ is a finite element space generated by a triangulation and finite elements satisfying **(A1)**,**(A2)** and **(A3)**.
We first provide an estimation of the **global interpolation error**. Throughout the following, $C \in \mathbb{R}_+$ denotes a generic constant independent of the mesh size $h$ which is not necessarily the same at each occurrence.

**Theorem 4.6 Estimation of the global interpolation error**

Assume that **(A1)**,**(A2)** and **(A3)** are satisfied and that there exist $k, \ell \in \mathbb{N}_0, \ell \le k$, such that the mapping

$$H^{k+1}(\hat{K}) \to C^s(\hat{K}) \tag{4.35}$$

is a continuous inclusion and

$$P_k(\hat{K}) \subset \hat{P}_{\hat{K}} \subset H^\ell(\hat{K}) , \tag{4.36}$$

where $s \in \mathbb{N}_0$ in (4.35) stands for the largest order of partial derivatives occurring in the definition of the set $\hat{\Sigma}_{\hat{K}}$ of degrees of freedom.

Then, the global interpolation operator $\Pi_h : H^{k+1}(K) \cap V \to V_h$ is well defined. If $v \in H^{k+1}(K) \cap V$, for $0 \le m \le \min(1, \ell)$ there holds

$$\|v - \Pi_h v\|_{m,\Omega} \ \le \ C \ h^{k+1-m} \ |v|_{k+1,\Omega} \ , \tag{4.37}$$

whereas for $2 \le m \le \min(k+1, \ell)$ we obtain

$$\Big( \sum_{K \in \mathcal{T}_h} \|v - \Pi_K v\|^2_{m,K} \Big)^{1/2} \ \le \ C \ h^{k+1-m} \ |v|_{k+1,\Omega} \ . \tag{4.38}$$

**Proof.** In view of Corollary 4.1, for the local interpolation error we get

$$\|v - \Pi_K v\|_{m,K} \ \le \ C \ h_K^{k+1-m} \ |v|_{k+1,K} \quad , \quad 0 \le m \le \min(k+1, \ell) \ .$$

Observing $(\Pi_h v)|_K = \Pi_K v, K \in \mathcal{T}_h$, and $h_K \le h, K \in \mathcal{T}_h$,, it follows that

$$\Big( \sum_{K \in \mathcal{T}_h} \|v - \Pi_K v\|^2_{m,K} \Big)^{1/2} \ \le \ C \ h^{k+1-m} \ \Big( \sum_{K \in \mathcal{T}_h} |v|^2_{k+1,K} \Big)^{1/2} \ =$$

$$= \ C \ h^{k+1-m} \ |v|_{k+1,\Omega} \quad , \quad 0 \le m \le \min(k+1, \ell) \ ,$$

which proves (4.38).
As far as the proof of (4.37) is concerned, for $m = 0$ and $m = 1$ (provided $\ell \ge 1$), we have $V_h \subset H^m(\Omega)$ and hence,

$$\Big( \sum_{K \in \mathcal{T}_h} \|v - \Pi_K v\|^2_{m,K} \Big)^{1/2} \ = \ \|v - \Pi_h v\|_{m,\Omega} \ . \quad \square$$

**Theorem 4.7 A priori estimate in the $H^1(\Omega)$-norm**

Suppose that the assumptions **(A1)**,**(A2)** and **(A3)** hold true and that there exists $k \in \mathbb{N}_0$ such that the mapping

$$H^{k+1}(\hat{K}) \to C^s(\hat{K}) \tag{4.39}$$

is a continuous inclusion and

$$P_k(\hat{K}) \subset \hat{P}_{\hat{K}} \subset H^1(\hat{K}) \ , \tag{4.40}$$

where $s \in \mathbb{N}_0$ in (4.65) stands for the largest order of partial derivatives occurring in the definition of the set $\hat{\Sigma}_{\hat{K}}$ of degrees of freedom.
Suppose further that the solution $u \in V$ of the variational equation under consideration satisfies the regularity assumption

$$u \in V \cap H^{k+1}(\Omega) \ . \tag{4.41}$$

Then, if $u_h \in V_h$ is the **conforming $C^0$-approximation** of $u$, for the **global discretization error** we have the following **a priori estimate**

$$\|u - u_h\|_{1,\Omega} \ \le \ C \ h^k \ |u|_{k+1,\Omega} \ . \tag{4.42}$$

**Proof.** The proof is an immediate consequence of Céa's lemma and Theorem 4.6. □

For so-called **(k+1)-regular** problems, we obtain an a priori estimate in terms of the data of the problem.

**Definition 4.2 k-regular problems**

Under the assumptions of the Lax-Milgram lemma, consider the variational equation

$$a(u,v) \ = \ \ell(v) \quad , \quad V \ , \tag{4.43}$$

where

$$\ell(v) \ := \ (f,v)_{0,\Omega} \quad , \quad f \in L^2(\Omega) \ . \tag{4.44}$$

Then, (4.43) is said to be **k-regular , k ≥ 2**, if $u \in V \cap H^k(\Omega)$ and there exists a constant $C_{reg} \in \mathbb{R}_+$ such that

$$\|u\|_{k,\Omega} \ \leq \ C_{reg} \ \|f\|_{0,\Omega} \ . \tag{4.45}$$

**Corollary 4.2 A priori estimate in the $H^1(\Omega)$-norm**

Let the assumptions of Theorem 4.7 be satisfied and suppose that the variational problem is (k+1)-regular. Then, the global discretization error satisfies

$$\|u - u_h\|_{1,\Omega} \ \leq \ C \ h^1 \ \|f\|_{0,\Omega} \ . \tag{4.46}$$

**Proof.** The proof follows readily from Theorem 4.7 and the (k+1)-regularity of the variational equation. □

## 4.4 A priori estimate in the $L^2(\Omega)$-norm

Theorem 4.7 shows that under sufficient regularity of the solution of the variational problem, the global discretization error $u-u_h$ is of order $O(h^k)$ with respect to the $\| \cdot \|_{1,\Omega}$-norm. Since $H^1(\Omega)$ is continuously imbedded in $L^2(\Omega)$, i.e., $\|v\|_{0,\Omega} \leq \|v\|_{1,\Omega}, v \in H^1(\Omega)$, we also have $\|u - u_h\|_{0,\Omega} = O(h^k)$. However, Theorem 4.6 tells us that under the same regularity assumption the global interpolation error $\|\Pi_h u - u\|_{0,\Omega}$ is of order $O(h^{k+1})$. As we will see in this section, under some additional assumption on the associated adjoint problem, this order also holds true for the global discretization error in the $L^2(\Omega)$-norm, a result which is known as the **quasi-optimality of the Galerkin approximation**.

The result can be shown by a **duality argument** which will be provided in an **abstract setting**.

Let $(H, (\cdot,\cdot)_H)$ and $(V, (\cdot,\cdot)_V)$ be **Hilbert spaces** such that $V$ is continuously and densely imbedded in $H$, i.e., the mapping

$$V \;\rightarrow\; H \tag{4.47}$$

is a **continuous imbedding** and

$$\overline{V}^{\|\cdot\|_H} \;=\; H \; . \tag{4.48}$$

Identifying $H$ with its dual, we may interpret $H$ as a subspace of $V^*$. Indeed, if $f \in H$, then (4.47) yields

$$|(f,v)_H| \;\leq\; \|f\|_H \|v\|_H \;\leq\; \|f\|_H \|v\|_V \quad , \quad v \in V \; ,$$

i.e., the linear functional $\tilde{f} : v \in V \longmapsto \tilde{f}(v) := (f,v)_H$ is bounded and thus $\tilde{f} \in V^*$.
Moreover, the mapping $f \in H \longmapsto \tilde{f} \in V^*$ is injective: if $\tilde{f}(v) = (f,v)_H = 0, v \in V$, then (4.48) implies $\tilde{f}(v) = 0, v \in H$, whence $f = 0$.

We consider the **abstract variational equation**

$$a(u,v) \;=\; \ell(v) \quad , \quad v \in V \; , \tag{4.49}$$

where $a(\cdot,\cdot) : V \times V \rightarrow \mathbb{R}$ and $\ell \in V^*$. We assume $V_h \subset V$ and approximate (4.49) by

$$a(u_h,v_h) \;=\; \ell(v_h) \quad , \quad v_h \in V_h \; . \tag{4.50}$$

The following result establishes an upper bound for the error $u - u_h$ in the $\|\cdot\|_H$-norm, which is known as the **Lemma of Aubin-Nitsche**, sometimes also called **Nitsche's trick**.

**Theorem 4.8 Lemma of Aubin-Nitsche**

Assume that $(V, (\cdot,\cdot)_V)$ is a Hilbert space, $a(\cdot,\cdot) : V \times V \rightarrow \mathbb{R}$ a bounded, $V$-elliptic bilinear form and $\ell \in V^*$. Suppose further that $(H, (\cdot,\cdot)_H)$ is a Hilbert space satisfying (4.47),(4.48) and $V_h \subset V$. Moreover, let $u \in V$ and $u_h \in V_h$ be the unique solutions of (4.49) and (4.50), respectively. Then there holds:

**(i)** For any $g \in H$, the **adjoint variational problem**

$$a(v,\varphi_g) \;=\; (g,v)_H \quad , \quad v \in V \tag{4.51}$$

admits a unique solution $\varphi_g \in V$.

**(ii)** The **global discretization error** $u - u_h$ satisfies

$$\|u - u_h\|_H \;\leq\; C\, \|u - u_h\|_V \left( \sup_{g \in H} \frac{1}{\|g\|_H} \inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_V \right) , \tag{4.52}$$

where the constant $C$ is an upper bound for the bilinear form $a(\cdot,\cdot)$.

**Proof.** Since $(g,v)_H = g(v), g \in H \subset V^*$, the Lemma of Lax-Milgram guarantees the existence and uniqueness of a solution of the adjoint problem (4.51).
Moreover, choosing $v = u - u_h \in V$ in (4.51), we have

$$a(u - u_h, \varphi_g) \ = \ (g, u - u_h)_H \ . \tag{4.53}$$

On the other hand, choosing $v = v_h = \varphi_h \in V_h$ in (4.49),(4.50) and subtracting (4.50) from (4.49), we obtain

$$a(u - u_h, \varphi_h) \ = \ 0 \quad , \quad \varphi_h \in V_h \ . \tag{4.54}$$

Subtracting (4.54) from (4.53), it follows that

$$a(u - u_h, \varphi_g - \varphi_h) \ = \ (g, u - u_h)_H \ ,$$

whence

$$|(g, u - u_h)_H| \ \leq \ C \ \|u - u_h\|_V \ \inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_V \ . \tag{4.55}$$

Finally, using

$$\|u - u_h\|_H \ = \ \sup_{g \in H, g \neq 0} \frac{|(g, u - u_h)_H|}{\|g\|_H}$$

in (4.55), gives the assertion. □

The **quasi-optimal a priori error estimate** in the $L^2(\Omega)$-norm is a direct consequence of the Lemma of Aubin-Nitsche.

**Theorem 4.9 A priori error estimate in the $L^2(\Omega)$-norm**

Assume that the **adjoint problem** (4.51) is **(k+1)-regular** and let the assumptions of Theorem 4.7 be satisfied. Then, if $u_h \in V_h$ is the **conforming $C^0$-approximation** of $u$, for the **global discretization error** we have the following **a priori estimate**

$$\|u - u_h\|_{0,\Omega} \ \leq \ C \ h^{k+1} \ |u|_{k+1,\Omega} \ . \tag{4.56}$$

**Proof.** The proof follows from the Lemma of Aubin-Nitsche for $H = L^2(\Omega)$. In particular, in view of Theorem 4.6 and the (k+1)-regularity of the adjoint problem, we have

$$\inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_{1,\Omega} \ \leq \ \|\varphi_g - \Pi_h \varphi_g\|_{1,\Omega} \ \leq \ C \ h \ \|\varphi_g\|_{k+1,\Omega} \ \leq \ C \ h \ \|g\|_{0,\Omega} \ .$$

Hence, (4.53) implies

$$\|u - u_h\|_{0,\Omega} \ \leq \ C \ h \ \|u - u_h\|_{1,\Omega} \ ,$$

and Theorem 4.7 allows a further estimation of the right-hand side which yields (4.56). □

## 4.5 Inverse estimates

In Theorem 4.6, the error in the approximation of a Sobolev space function by its finite element interpolate is measured in a coarser Sobolev space norm than the Sobolev space norm of the given function. In general, the reverse does not hold true, i.e., it is not possible to estimate finer Sobolev space norms from above by coarser ones. However, for finite element functions, such estimates can be established which are known as **inverse estimates**.

**Theorem 4.10 Inverse estimates**

Assume that **(A1)**,**(A2)** and **(A3)** are satisfied and that there exist $k \in \mathbb{N}_0$ such that the mapping

$$H^{k+1}(\hat{K}) \to C^s(\hat{K}) \tag{4.57}$$

is a continuous inclusion and

$$P_k(\hat{K}) \subset \hat{P}_{\hat{K}} \subset H^{k+1}(\hat{K}) \ , \tag{4.58}$$

where $s \in \mathbb{N}_0$ in (4.57) stands for the largest order of partial derivatives occurring in the definition of the set $\hat{\Sigma}_{\hat{K}}$ of degrees of freedom.
Further, assume that $(V_h)_{h\in\mathcal{H}}$ is the associated family of conforming finite element spaces. Then, there exists a constant $C(\sigma, k, \ell) > 0$ such that for $0 \le k+1 \le \ell$ the following **inverse estimates** hold true

$$\|v_h\|_{\ell,\Omega} \ \le \ C(\sigma,k,\ell) \ h^{k+1-\ell} \ \|v_h\|_{k+1,\Omega} \quad , \quad v_h \in V_h \ . \tag{4.59}$$

**Proof.** In a first step, for the reference element we prove

$$|\hat{v}|_{\ell,\hat{K}} \ \le \ C \ |\hat{v}|_{k+1,\hat{K}} \quad , \quad \hat{v} \in \hat{P}_{\hat{K}} \ . \tag{4.60}$$

Assume $\hat{P}_{\hat{K}} = \text{span}(\hat{p}_1, ..., \hat{p}_{n_k})$ and let $(\hat{f}_i)_{i=1}^{n_k}$ be a basis of the dual space, i.e.,

$$\begin{aligned} \hat{f}_i \ : \ \hat{P}_{\hat{K}} \quad &\to \quad \mathbb{R} \quad , \quad 1 \le i \le n_k \ , \\ \hat{p}_j \quad &\longmapsto \quad \hat{f}_i(\hat{p}_j) = \delta_{ij} \quad , \quad 1 \le i,j \le n_k \ . \end{aligned}$$

We note that the $\|\cdot\|_{k+1,\hat{K}}$-norm and the norm

$$|||\hat{v}|||_{k+1,\hat{K}} \ := \ |\hat{v}|_{k+1,\hat{K}} \ + \ \sum_{i=1}^{n_k} |\hat{f}_i(\hat{v})|$$

are equivalent. Furthermore, due to the finite dimension of $V_h$, the norms $\|\cdot\|_{k+1,\hat{K}}$ and $\|\cdot\|_{\ell,\hat{K}}$ are equivalent as well.

It follows that

$$\begin{aligned} |\hat{v}|_{\ell,\hat{K}} &\leq |\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v}|_{\ell,\hat{K}} + \underbrace{|\hat{\Pi}_{\hat{K}}\hat{v}|_{\ell,\hat{K}}}_{=\,0} \leq \\ &\leq \|\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v}\|_{\ell,\hat{K}} \leq C \, \|\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v}\|_{k+1,\hat{K}} = \\ &= C\Big( |\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v}|_{k+1,\hat{K}} + \underbrace{\sum_{i=1}^{n_k} |\hat{f}_i(\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v})|}_{=\,0} \Big) \leq \\ &\leq C \, \Big( |\hat{v}|_{k+1,\hat{K}} + \underbrace{|\hat{\Pi}_{\hat{K}}\hat{v}|_{k+1,\hat{K}}}_{=\,0} \Big) , \end{aligned}$$

which proves (4.60).
The assertion can then be deduced by a **standard scaling argument**

$$\begin{aligned} |v|_{\ell,K} &\leq C \, \|B_K^{-1}\|^\ell \, |\det(B_k)|^{1/2} \, |\hat{v}|_{\ell,\hat{K}} \leq \\ &\leq C \, \|B_K^{-1}\|^\ell \, |\det(B_k)|^{1/2} \, |\hat{v}|_{k+1,\hat{K}} \leq \\ &= C \, \|B_K^{-1}\|^\ell \, \|B_K\|^{k+1} |v|_{k+1,K} \leq \\ &\leq C \, h_K^{k+1-\ell} \, |v|_{k+1,K} \, . \quad \square \end{aligned}$$

We now consider approximation properties and inverse estimates in a somewhat more **general framework**:
Let $V$ and $H$ be Hilbert spaces such that $V$ is continuously imbedded in $H$ and assume that $(V_h)_{h\in\mathcal{H}}$ is a family of finite dimensional subspaces $V_h \subset V$ such that for some $\alpha, \beta \in \mathbb{N}_0$ the **approximation property**

(4.61) $$\inf_{v_h \in V_h} \|u - v_h\|_H \leq C \, C \, h^\alpha \, \|u\|_V \quad , \quad u \in V$$

and the **inverse estimate**

(4.62) $$\|v_h\|_V \leq C \, h^{-\beta} \, \|v_h\|_H \quad , \quad v_h \in V_h$$

hold true.

**Definition 4.3 Optimality of estimates**

The approximation property (4.61) and the inverse estimate (4.62) are called **optimal**, if $\beta = \alpha$ holds true.

**Theorem 4.11 Quasi-optimality of the estimates**

Assume that $V$ is compactly imbedded in $H$ and that the approximation property (4.61) and the inverse estimate (4.62) hold true. Then, there holds

(4.63) $$\beta \geq \alpha \, .$$

**Proof.** The proof is by a **contradiction argument**: Assume $\alpha > \beta$. Then, there exist $1 < \gamma < 2$ and a sequence of nested spaces $(V_{h_n})_{n\in\mathbb{N}_0}$

$$V_{h_0} \subset V_{h_1} \subset V_{h_2} \subset \ldots$$

such that

$$\begin{aligned} \inf_{n\in\mathbb{N}_0} \|u - v_{h_n}\|_H &\leq 2^{-\gamma n} \|u\|_V , \\ \|v_{h_n}\|_V &\leq 2^n \|v_{h_n}\|_H . \end{aligned}$$

Now, choose $m \in \mathbb{N}$ such that

$$2^{-(\gamma-1)m} < (1 - \frac{2^{-\gamma-1}}{5} .$$

Since $V$ is compactly imbedded in $H$, there exists $u \in V$ with

(4.64) $$\|u\|_V = 1 \quad , \quad \|u\|_H < \varepsilon := 2^{-\gamma m} .$$

Define $w_{h_n} \in V_{h_n}$ according to

$$w_{h_n} := \begin{cases} v_{h_n} & , \quad n \leq m \\ v_{h_n} - v_{h_{n-1}} & , \quad n > m \end{cases} .$$

It follows that

$$\|w_{h_m}\|_H \leq \|u - w_{h_m}\|_H + \|u\|_H \leq 2 \cdot 2^{-\gamma m}$$

and for $n > m$

$$\begin{aligned} \|w_{h_n}\|_H &= \|v_{h_n} - v_{h_n} \pm u\|_H \leq \\ &\leq \|u - v_{h_n}\|_H + \|u - v_{h_{n-1}}\|_H \leq \\ &\leq 2^{-\gamma n} + 2^{-\gamma(n-1)} = (1 + 2^\gamma)\, 2^{-\gamma n} \leq \\ &\leq 5 \cdot 2^{-\gamma n} . \end{aligned}$$

Consequently,

$$\|w_{h_n}\|_V \leq 2^n \|w_{h_n}\|_H \leq 5 \cdot 2^{-(\gamma-1)n} \quad , \quad n > m ,$$

and hence,

$$\begin{aligned} \|u\|_V &= \|\sum_{n=m}^{\infty} w_{h_n}\|_V \leq \sum_{n=m}^{\infty} \|w_{h_n}\|_V \leq \\ &\leq 5 \left( \sum_{n=0}^{\infty} 2^{-(\gamma-1)n} - \sum_{n=0}^{m} 2^{-(\gamma-1)n} \right) = \\ &= 5 \left( \frac{1}{1 - 2^{-(\gamma-1)}} - \frac{1 - 2^{-(\gamma-1)m}}{1 - 2^{-(\gamma-1)}} \right) = \\ &= \frac{5 \cdot 2^{-(\gamma-1)m}}{1 - 2^{-(\gamma-1)}} < 1 , \end{aligned}$$

which clearly contradicts (4.64). □

## 4.6 A priori error estimate in the $L^\infty$-norm

Quasi-optimal a priori error estimates in the $L^\infty$-norm can be derived by using **weighted norms in Sobolev spaces** (cf., e.g., [?]). Instead, we will provide a much simpler, **non optimal estimate**.

**Theorem 4.12 A priori error estimate in the $L^\infty$-norm**
Suppose that the assumptions **(A1)**,**(A2)** and **(A3)** hold true and that there exists $k \in \mathbb{N}_0$ such that the mapping

$$H^{k+1}(\hat{K}) \to C^s(\hat{K}) \tag{4.65}$$

is a continuous inclusion and

$$P_k(\hat{K}) \subset \hat{P}_{\hat{K}} \subset H^1(\hat{K}) \ , \tag{4.66}$$

where $s \in \mathbb{N}_0$ in (4.65) stands for the largest order of partial derivatives occurring in the definition of the set $\hat{\Sigma}_{\hat{K}}$ of degrees of freedom.
Suppose further that the solution $u \in V$ of the variational equation under consideration satisfies the regularity assumption

$$u \in V \cap H^{k+1}(\Omega) \ . \tag{4.67}$$

Then, if $u_h \in V_h$ is the **conforming $C^0$-approximation** of $u$, for the **global discretization error** we have the following **a priori estimate**

$$\|u - u_h\|_{\infty,\Omega} \ \leq \ C \ h^k \ |u|_{k+1,\Omega} \ . \tag{4.68}$$

**Proof.** Using the global interpolation operator $\Pi_h$, we split the error according to

$$\|u - u_h\|_{\infty,\Omega} \ \leq \ \|u - \Pi_h u\|_{\infty,\Omega} \ + \ \|\Pi_h u - u_h\|_{\infty,\Omega} \ ,$$

and estimate both parts on the right-hand side separately.

**(i)** Estimation of $\|u - \Pi_h u\|_{\infty,\Omega}$

Using the affine equivalence of the finite elements, for $\hat{v} \in H^{k+1}(\hat{K})$, the **Bramble-Hilbert lemma** yields

$$\|\hat{v} - \hat{\Pi}_{\hat{K}}\hat{v}\|_{\infty,\hat{K}} \ \leq \ C \ |\hat{v}|_{k+1,\hat{K}} \ .$$

Then, by the **standard scaling argument**

$$\|u - \Pi_K u\|_{\infty,K} \ \leq \ \|\hat{u} - \hat{\Pi}_{\hat{K}}\hat{u}\|_{\infty,\hat{K}} \ \leq \ C \ |\hat{u}|_{k+1,\hat{K}} \ \leq \ C \ h^k \ |u|_{k+1,K} \ ,$$

whence

$$\begin{aligned} \|u - \Pi_h u\|_{\infty,\Omega} \ &= \ \max_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{\infty,K} \ \leq \\ &\leq \ C \ h^k \ \max_{K \in \mathcal{T}_h} |u|_{k+1,K} \ \leq \ C \ h^k \ |u|_{k+1,\Omega} \ . \end{aligned} \tag{4.69}$$

**(ii)** Estimation of $\|\Pi_h u - u_h\|_{\infty,\Omega}$

By the **Sobolev imbedding theorem** and the **inverse estimate**, we obtain

$$\|u_h - \Pi_h u\|_{\infty,\Omega} \ \leq \ C \ \|u_h - \Pi_h u\|_{1,\Omega} \ \leq \ C \ h^{-1} \ \|u_h - \Pi_h u\|_{0,\Omega} \ .$$

Taking advantage of the **a priori error estimate in the $L^2$-norm** and the **approximation property**, it follows that

$$\|u_h - \Pi_h u\|_{0,\Omega} \ \leq \ C \ \|u_h - u\|_{0,\Omega} \ + \ \|u - \Pi_h u\|_{0,\Omega} \leq \ C \ h^{k+1} \ |u|_{k+1,\Omega} \ ,$$

whence

$$\|u_h - \Pi_h u\|_{0,\Omega} \ \leq \ C \ h^k \ |u|_{k+1,\Omega} \ . \tag{4.70}$$

Combining (4.69) and (4.70) gives the assertion. □

**Remark 4.1 Quasi-optimal a priori estimate**

Assuming $u \in V \cap W^{2,\infty}(\Omega)$, it can be shown that

$$\|u - u_h\|_{\infty,\Omega} \ \leq \ C \ h^2 \ |\log h|^{3/2} \ \|D^2 u\|_{\infty,\Omega} \ . \tag{4.71}$$

Note that this estimate is **optimal** with respect to the order of $h$, i.e., the log-term in (4.71) can not be removed.