

## Chapter 5 A priori error estimates for nonconforming finite element approximations

### 5.1 Strang's first lemma

We consider the variational equation

$$(5.1) \quad a(u, v) = \ell(v) \quad , \quad v \in V \subset H^1(\Omega) \quad ,$$

and assume that the conditions of the Lax-Milgram lemma are satisfied so that (5.1) admits a unique solution  $u \in V$ .

We assume that  $\mathcal{H}$  is a null sequence of positive real numbers and  $(V_h)_{h \in \mathcal{H}}$  an associated **family of conforming finite element spaces**  $V_h \subset V, h \in \mathcal{H}$ . We approximate the bilinear form  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  and the functional  $\ell(\cdot) : V \rightarrow \mathbb{R}$  in (5.1) by bounded bilinear forms and bounded linear functionals

$$(5.2) \quad a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R} \quad , \quad \ell_h(\cdot) : V_h \rightarrow \mathbb{R} \quad , \quad h \in \mathcal{H}$$

and consider the finite dimensional variational equations

$$(5.3) \quad a_h(u_h, v_h) = \ell_h(v_h) \quad , \quad v_h \in V_h \quad , \quad h \in \mathcal{H} \quad .$$

#### Definition 5.1 Uniform $V_h$ -ellipticity

The sequence  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  of approximate bilinear forms  $a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$  is called **uniformly  $V_h$ -elliptic**, if there exists a positive constant  $\tilde{\alpha}$  such that uniformly in  $h \in \mathcal{H}$

$$(5.4) \quad a_h(u_h, u_h) \geq \tilde{\alpha} \|u_h\|_V^2 \quad , \quad u_h \in V_h \quad .$$

Under the assumption of uniform  $V_h$ -ellipticity, the variational equations (5.3) admit unique solutions  $u_h \in V_h$ . The following result, known as **Strang's first lemma**, can be viewed as a generalization of Céa's lemma.

#### Theorem 5.1 Strang's first lemma

Assume that  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  is a **uniformly  $V_h$ -elliptic** family of bilinear forms and denote by  $u \in V$  and  $u_h \in V_h, h \in \mathcal{H}$  the unique solutions of (5.1) and (5.3), respectively. then, there exists a constant  $C \in \mathbb{R}_+$ , independent of  $h \in \mathcal{H}$  such that

$$(5.5) \quad \|u - u_h\|_V \leq \left( \inf_{v_h \in V_h} (\|u - v_h\|_V + \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V}) + \sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V} \right) \quad , \quad h \in \mathcal{H} \quad .$$

**Proof.** Taking advantage of the uniform  $V_h$ -ellipticity, for  $v_h \in V_h$  we obtain

$$\begin{aligned} \tilde{\alpha} \|u_h - v_h\|_V^2 &\leq a_h(u_h - v_h, u_h - v_h) \pm a(u - v_h, u_h - v_h) = \\ &= a(u - v_h, u_h - v_h) + \left( a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h) \right) + \\ &\quad + \left( \ell_h(u_h - v_h) - \ell(u_h - v_h) \right). \end{aligned}$$

Using the boundedness of the bilinear form, i.e.,  $|a(u, v)| \leq M \|u\|_V \|v\|_V$ , and dividing the previous inequality by  $\|u_h - v_h\|_V$ , it follows that

$$\begin{aligned} \tilde{\alpha} \|u_h - v_h\|_V^2 &\leq M \|u_h - v_h\|_V + \frac{|a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h)|}{\|u_h - v_h\|_V} + \\ &\quad + \frac{|\ell_h(u_h - v_h) - \ell(u_h - v_h)|}{\|u_h - v_h\|_V} \leq \\ &\leq M \|u_h - v_h\|_V + \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V} + \\ &\quad + \sup_{w_h \in V_h} \frac{|\ell_h(w_h) - \ell(w_h)|}{\|w_h\|_V}. \end{aligned}$$

Using the triangle inequality

$$\|u - u_h\|_V \leq \|u - v_h\|_V + \|u_h - v_h\|_V$$

and the previous inequality, we deduce (5.5).  $\square$

Strang's first lemma shows that the upper bound for the global discretization error consists of two parts: the **approximation error**

$$(5.6) \quad \inf_{v_h \in V_h} \|u - v_h\|_V$$

and the **consistency errors**

$$(5.7) \quad \inf_{v_h \in V_h} \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V},$$

$$(5.8) \quad \sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V}.$$

As an application of **Strang's first lemma**, in the subsequent section we will study the effect of **numerical integration** in the finite element approximation of second order elliptic boundary value problems.

## 5.2 Numerical integration

### 5.2.1 Construction of quadrature formulas

We consider the **model variational equation**

$$(5.9) \quad a(u, v) = \ell(v) \quad , \quad v \in V := H_0^1(\Omega) \quad ,$$

where  $\Omega \subset \mathbb{R}^d$  is supposed to be a polygonal resp. polyhedral domain and the bilinear form  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  and the functional  $\ell(\cdot) : V \rightarrow \mathbb{R}$  are given according to

$$(5.10) \quad a(u, v) := \int_{\Omega} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \quad ,$$

and

$$(5.11) \quad \ell(v) := \int_{\Omega} f v dx \quad .$$

Here,  $f \in L^2(\Omega)$  and  $a_{ij} \in C(\overline{\Omega})$ ,  $1 \leq i, j \leq d$ , satisfying

$$(5.12) \quad \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^d \xi_i^2 \quad , \quad x \in \overline{\Omega} \quad , \quad \alpha > 0 \quad .$$

We approximate (5.9) using a finite element space  $V_h$  generated by a triangulation  $\mathcal{T}_h$  and finite elements  $(K, P_K, \Sigma_K)$ ,  $K \in \mathcal{T}_h$ , satisfying the assumptions **(A1)**, **(A2)**, **(A3)** of section 4.3 and  $\hat{P}_{\hat{K}} \subset H^1(\hat{K})$  which implies **conformity**, i.e.,  $V_h \subset V$ .

We approximate the bilinear form (5.10) and the functional (5.11) by evaluating the integrals by means of appropriate **quadrature formulas** with respect to the individual elements  $K \in \mathcal{T}_h$  of the triangulation. Taking into account the **affine equivalence** of the finite elements with respect to a **reference element**  $\hat{K}$

$$\begin{aligned} F_K : \hat{K} &\rightarrow K \\ \hat{x} &\rightarrow F_K(\hat{x}) = B_K \hat{x} + b_K \quad , \quad K \in \mathcal{T}_h \quad , \end{aligned}$$

due to the **transformation of variables formula**

$$(5.13) \quad \int_K \varphi(x) dx = \det(B_K) \int_{\hat{K}} \hat{\varphi}(\hat{x}) d\hat{x} \quad , \quad \hat{\varphi} := \varphi \circ F_K \quad ,$$

a quadrature formula for an integral over the reference element  $\hat{K}$  induces a suitable quadrature formula for the associated integral over the

actual element  $K = F_K(\hat{K})$  of  $\mathcal{T}_h$ .

In particular, we consider the **reference quadrature formula**

$$(5.14) \quad \hat{Q}_{\hat{K}}(\hat{\varphi}) := \sum_{\ell=1}^L \hat{\omega}_{\ell, \hat{K}} \hat{\varphi}(\hat{b}_{\ell, \hat{K}}),$$

with **weights**  $\hat{\omega}_{\ell, \hat{K}}$ ,  $1 \leq \ell \leq L$ , and **nodes**  $\hat{b}_{\ell, \hat{K}} \in \hat{K}$ ,  $1 \leq \ell \leq L$ .  
We refer to

$$(5.15) \quad \hat{E}_{\hat{K}}(\hat{\varphi}) := \int_{\hat{K}} \hat{\varphi}(\hat{x}) d\hat{x} - \hat{Q}_{\hat{K}}(\hat{\varphi})$$

as the associated **quadrature error functional**.

### **Lemma 5.1 Quadrature formulas in the FEM**

Assume that  $\hat{Q}_{\hat{K}}$  is a quadrature formula with respect to the reference element  $\hat{K}$  as given by (5.14). Then, for the affine equivalent element  $K := F_K(\hat{K})$  we obtain a quadrature formula according to

$$(5.16) \quad Q_K(\varphi) := \sum_{i=1}^L \omega_{i,K} \varphi(b_{i,K}),$$

$$\omega_{\ell,K} := \det(B_K) \hat{\omega}_{\ell, \hat{K}}, \quad b_{\ell,K} := F_K(\hat{b}_{\ell, \hat{K}}), \quad 1 \leq \ell \leq L.$$

Moreover, referring to

$$(5.17) \quad E_K(\varphi) := \int_K \varphi(x) dx - Q_K(\varphi)$$

as the related quadrature error functional, we have

$$(5.18) \quad E_K(\varphi) = \det(B_K) \hat{E}_{\hat{K}}(\hat{\varphi}).$$

**Proof.** The proof is a direct consequence of (5.13).  $\square$

Based on the quadrature formula (5.16), we define the **approximate bilinear forms**  $a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$  by means of

$$(5.19) \quad a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^L \omega_{\ell,K} \sum_{u,j=1}^d (a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j})(b_{\ell,K}),$$

and the **approximate linear functionals**  $\ell_h(\cdot) : V_h \rightarrow \mathbb{R}$  according to

$$(5.20) \quad \ell_h(v_h) := \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^L \omega_{\ell,K} (fv_h)(b_{\ell,K}).$$

Then, the **finite element approximation** of the model problem (5.9) amounts to the computation of  $u_h \in V_h, h \in \mathcal{H}$ , as the solution of the approximate variational equation

$$(5.21) \quad a_h(u_h, v_h) = \ell_h(v_h), \quad v_h \in V_h.$$

We note that according to the Lax-Milgram lemma, (5.21) admits a unique solution, if the bilinear form  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  is bounded and  $V_h$ -elliptic.

We first derive **sufficient conditions** for the **uniform  $V_h$ -ellipticity** of the family  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  of approximate bilinear forms.

**Theorem 5.2 Sufficient conditions for uniform  $V_h$ -ellipticity**

Let  $\hat{Q}_{\hat{K}}$  be a quadrature formula with respect to the reference element  $\hat{K}$  with positive weights  $\hat{\omega}_{\ell, \hat{K}}, 1 \leq \ell \leq L$ , and the property that there exists  $q \in \mathbb{N}$  such that

- $\hat{P}_{\hat{K}} \subset P_q(\hat{K})$  ,
- (i) the quadrature formula is exact for polynomials  $\hat{p} \in P_{2q-2}(\hat{K})$ ,  
or
- (ii) the union of nodes  $\bigcup_{\ell=1}^L \{\hat{b}_{\ell, \hat{K}}\}$  contains a  $P_{q-1}(\hat{K})$ -unisolvent subset.

Then, there exists a positive constant  $\tilde{\alpha}$ , independent of  $h \in \mathcal{H}$ , such that

$$(5.22) \quad a_h(v_h, v_h) \geq \tilde{\alpha} |v_h|_{1, \Omega}^2, \quad v_h \in V_h, \quad h \in \mathcal{H}.$$

**Proof.** Observing  $v_h|_K = p_K \in P_K$  and using the ellipticity condition (5.12), we find

$$(5.23) \quad \begin{aligned} & \sum_{\ell=1}^L \omega_{\ell, K} \sum_{i,j=1}^d (a_{ij} \frac{\partial v_h}{\partial x_i} \frac{\partial v_h}{\partial x_j})(b_{\ell, K}) = \\ & = \sum_{\ell=1}^L \omega_{\ell, K} \sum_{i,j=1}^d (a_{ij} \frac{\partial p_K}{\partial x_i} \frac{\partial p_K}{\partial x_j})(b_{\ell, K}) \geq \\ & \geq \alpha \sum_{\ell=1}^L \omega_{\ell, K} \underbrace{\sum_{i=1}^d \left( \frac{\partial p_K}{\partial x_i}(b_{\ell, K}) \right)^2}_{= \|Dp_K(b_{\ell, K})\|^2}. \end{aligned}$$

Now, observing

$$D\hat{p}_{\hat{K}}(\hat{b}_{\ell, \hat{K}})\xi = Dp(b_{\ell, K})(B_K\xi) \quad , \quad 1 \leq \ell \leq L,$$

we have

$$\|D\hat{p}_{\hat{K}}(\hat{b}_{\ell,\hat{K}})\| \leq \|B_K\| \|Dp(b_{\ell,K})\| \quad , \quad 1 \leq \ell \leq L \quad ,$$

and hence, using Theorem 4.3, we get

$$(5.24) \quad \sum_{\ell=1}^L \omega_{\ell,K} \underbrace{\sum_{i=1}^d \left(\frac{\partial p_K}{\partial x_i}(b_{\ell,K})\right)^2}_{= \|Dp_K(b_{\ell,K})\|^2} \geq \|B_K\|^{-2} \sum_{\ell=1}^L \omega_{\ell,K} \sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\right)^2$$

$$= \det(B_K) \|B_K\|^{-2} \sum_{\ell=1}^L \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\right)^2 .$$

(i) Let us first assume that the quadrature formula  $\hat{Q}_{\hat{K}}$  is exact for polynomials  $\hat{p} \in P_{2q-2}(\hat{K})$ . Since

$$\sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}\right)^2 \in P_{2q-2}(\hat{K}) \quad ,$$

we then have

$$(5.25) \quad |\hat{p}_{\hat{K}}|_{1,\hat{K}}^2 = \int_{\hat{K}} \sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}\right)^2 d\hat{x} = \sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\right)^2 .$$

Inserting (5.25) into (5.24) and using again Theorem 4.3, it follows that

$$(5.26) \quad \sum_{\ell=1}^L \omega_{\ell,K} \sum_{i=1}^d \left(\frac{\partial p_K}{\partial x_i}(b_{\ell,K})\right)^2 \geq$$

$$\geq \det(B_K) \|B_K\|^{-2} \sum_{\ell=1}^L \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^d \left(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\right)^2 =$$

$$= \det(B_K) \|B_K\|^{-2} |\hat{p}_{\hat{K}}|_{1,\hat{K}}^2 \geq$$

$$\geq \left(\|B_K\| \|B_K^{-1}\|\right)^{-2} |p_K|_{1,K}^2 .$$

Due to the shape regularity of  $\mathcal{T}_h, h \in \mathcal{H}$ ,

$$\|B_K\| \|B_K^{-1}\| \leq \frac{\hat{h}_{\hat{K}}}{\hat{\rho}_{\hat{K}}} \frac{h_K}{\rho_K} \leq C .$$

Combining (5.23),(5.24) and (5.26), we deduce the existence of a positive constant  $\tilde{\alpha}$ , independent of  $h \in \mathcal{H}$ , such that

$$(5.27) \quad \sum_{\ell=1}^L \omega_{\ell,K} \sum_{i,j=1}^d (a_{ij} \frac{\partial v_h}{\partial x_i} \frac{\partial v_h}{\partial x_j})(b_{\ell,K}) \geq \tilde{\alpha} |v_h|_{1,K}^2 \quad , \quad v_h \in V_h .$$

Forming the sum over all  $K \in \mathcal{T}_h$  in (5.27), gives the assertion.

(ii) What remains to be shown is that the assertion also holds true, if we assume that the union  $\bigcup_{\ell=1}^L \{\hat{b}_{\ell, \hat{K}}\}$  contains a  $P_{q-1}(\hat{K})$ -unisolvent subset. We claim that in this case (5.25) has to be replaced by

$$(5.28) \quad \sum_{i=1}^d \left( \frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell, \hat{K}}) \right)^2 \geq \hat{C} |\hat{p}_{\hat{K}}|_{1, \hat{K}}^2 ,$$

where  $\hat{C}$  is a positive constant. The proof then proceeds in the same way as before. In order to verify (5.28), it suffices to show that

$$\left( \sum_{\ell=1}^L \hat{\omega}_{\ell, \hat{K}} \sum_{i=1}^d \left( \frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell, \hat{K}}) \right)^2 \right)^{1/2}$$

provides a norm on the quotient space  $\hat{P}_{\hat{K}}/P_0(\hat{K})$ , since so does  $|\cdot|_{1, \hat{K}}$ , and we may conclude taking advantage of the equivalence of norms on finite dimensional spaces.

For that purpose, we assume

$$\sum_{\ell=1}^L \hat{\omega}_{\ell, \hat{K}} \sum_{i=1}^d \left( \frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell, \hat{K}}) \right)^2 = 0 .$$

Then, the positivity of the weights  $\hat{\omega}_{\ell, \hat{K}}, 1 \leq \ell \leq L$ , yields

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell, \hat{K}}) = 0 \quad , \quad 1 \leq i \leq d , \quad 1 \leq \ell \leq L .$$

But for each  $i \in \{1, \dots, d\}$

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i} \in P_{q-1}(\hat{K}) ,$$

and hence,

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i} \equiv 0 ,$$

since it vanishes on a  $P_{q-1}(\hat{K})$ -unisolvent subset.  $\square$

With Theorem 5.2 at hand, we may apply **Strang's first lemma**. If the solution  $u \in V$  of (5.9) satisfies  $u \in V \cap H^{k+1}(\Omega), k \in \mathbb{N}$ , for the **approximation error** we get

$$(5.29) \quad \inf_{v_h \in V_h} \|u - v_h\|_{1, \Omega} \leq \|u - \Pi_h u\|_{1, \Omega} \leq C h^k |u|_{k+1, \Omega} .$$

Therefore, we have to provide **sufficient conditions** ensuring that the **consistency error** does not deteriorate this order of convergence, i.e.,

we are looking for **consistency error estimates** of the form

$$(5.30) \quad \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V} \leq C(a_{ij}, u) h^k ,$$

$$(5.31) \quad \sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V} \leq C(\ell) h^k ,$$

where  $C(a_{ij}, u)$  and  $C(\ell)$  are positive constants, independent of  $h \in \mathcal{H}$ .

The following auxiliary result will be used to establish such consistency error estimates.

**Lemma 5.2 Generalized Leibniz formula**

Assume that  $\Omega \subset \mathbb{R}^d$  is a Lipschitz domain and  $u \in H^m(\Omega)$  and  $v \in W^{m, \infty}(\Omega)$  for some  $m \in \mathbb{N}_0$ . Then, there exists a positive constant  $C(m, d)$  such that

$$(5.32) \quad |uv|_{m, \Omega} \leq C(m, d) \sum_{j=0}^m |u|_{m-j, \Omega} |v|_{j, \infty, \Omega} .$$

**Proof.** The proof is left as an exercise.  $\square$

**Theorem 5.3 Consistency error estimate I**

Let  $\hat{Q}_{\hat{K}}$  be a quadrature formula with respect to the reference element  $\hat{K}$  with positive weights  $\hat{\omega}_{\ell, \hat{K}}, 1 \leq \ell \leq L$ , and the property that there exists  $k \in \mathbb{N}$  such that

$$(5.33) \quad \hat{P}_{\hat{K}} = P_k(\hat{K}) ,$$

$$(5.34) \quad \hat{E}_{\hat{K}}(\hat{\varphi}) = 0 , \quad \hat{\varphi} \in P_{2k-2}(\hat{K}) .$$

Then, there exists a positive constant  $C$ , independent of  $h \in \mathcal{H}$ , such that for all  $a \in W^{k, \infty}(\Omega)$  and all  $p, q \in P_k(K)$

$$(5.35) \quad |E_K(a \frac{\partial q}{\partial x_i} \frac{\partial p}{\partial x_j})| \leq C h_K^k \|a\|_{k, \infty, K} |p|_{1, K} \|q\|_{k, K} .$$

**Proof.** Since  $\frac{\partial q}{\partial x_i}, \frac{\partial p}{\partial x_j} \in P_{k-1}(K)$ , it suffices to prove (5.35) for

$$E_K(aww) , \quad a \in W^{k, \infty}(K) , \quad v, w \in P_{k-1}(K) .$$

In view of (5.18) in Lemma 5.1, we have

$$E_K(aww) = \det(B_K) \hat{E}_{\hat{K}}(\hat{a}\hat{v}\hat{w})$$

where

$$\hat{a} \in W^{k, \infty}(\hat{K}) , \quad \hat{v}, \hat{w} \in P_{k-1}(\hat{K}) .$$



(i) Since  $\hat{a}\hat{v} \in W^{k,\infty}(\hat{K})$ , we first provide an estimate for

$$\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w}) \quad , \quad \hat{\varphi} \in W^{k,\infty}(\hat{K}) \quad , \quad \hat{w} \in P_{k-1}(\hat{K}) .$$

By direct estimation and using the equivalence of norms on  $P_{k-1}(\hat{K})$ , we obtain

$$\begin{aligned} |\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})| &= \left| \int_{\hat{K}} \hat{\varphi}\hat{w} \, d\hat{x} - \sum_{\ell=1}^L \hat{\omega}_{\ell,\hat{K}}(\hat{\varphi}\hat{w})(\hat{b}_{\ell,\hat{K}}) \right| \leq \\ &\leq \hat{C} |\hat{\varphi}\hat{w}|_{0,\infty,\hat{K}} \leq \hat{C} |\hat{\varphi}|_{0,\infty,\hat{K}} |\hat{w}|_{0,\infty,\hat{K}} \leq \hat{C} \|\hat{\varphi}\|_{k,\infty,\hat{K}} |\hat{w}|_{0,\hat{K}} , \end{aligned}$$

which proves the continuity of the functional  $\hat{E}_{\hat{K}}(\hat{w}) : W^{k,\infty}(\hat{K}) \rightarrow \mathbb{R}$  given by  $(\hat{E}_{\hat{K}}(\hat{w}))(\hat{\varphi}) := \hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})$  with

$$(5.36) \quad \|\hat{E}_{\hat{K}}(\hat{w})\| \leq \hat{C} \|\hat{w}\|_{0,\hat{K}} .$$

Since  $P_{k-1}(\hat{K}) \subset \text{Ker}(\hat{E}_{\hat{K}}(\hat{w}))$ , using (5.36) in the **Bramble-Hilbert lemma**, we get

$$(5.37) \quad |\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})| \leq \hat{C} |\hat{\varphi}|_{k,\infty,\hat{K}} \|\hat{w}\|_{0,\hat{K}} .$$

(ii) We now consider the case  $\hat{\varphi} = \hat{a}\hat{v}$  where  $\hat{a} \in W^{k,\infty}(\hat{K})$  and  $\hat{v} \in P_{k-1}(\hat{K})$ . In view of the **generalized Leibniz formula** (cf. Lemma 5.2) and taking  $|\hat{v}|_{k,\infty,\hat{K}} = 0$  into account, we obtain

$$(5.38) \quad |\hat{E}_{\hat{K}}(\hat{a}\hat{v}\hat{w})| \leq \hat{C} \left( \sum_{j=1}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}} |\hat{v}|_{j,\hat{K}} \right) \|\hat{w}\|_{0,\hat{K}} .$$

Now, Theorem 4.3 tells us

$$(5.39) \quad |\hat{a}|_{k-j,\infty,\hat{K}} \leq \hat{C} h_K^{k-j} |a|_{k-j,\infty,K} \quad , \quad 0 \leq j \leq k-1 ,$$

$$(5.40) \quad |\hat{v}|_{j,\hat{K}} \leq \hat{C} |\det(B_K)|^{-1/2} h_K^j |v|_{j,K} \quad , \quad 0 \leq j \leq k-1 ,$$

$$(5.41) \quad \|\hat{w}\|_{0,\hat{K}} \leq \hat{C} |\det(B_K)|^{-1/2} \|w\|_{0,K} .$$

Using (5.38)-(5.41), we finally obtain

$$\begin{aligned} |E_K(avw)| &\leq C h_K^k \left( \sum_{j=0}^{k-1} |a|_{k-j,\infty,K} |v|_{j,K} \right) \|w\|_{0,K} \leq \\ &\leq C h_K^k \|a\|_{k,\infty,K} \|v\|_{k-1,K} \|w\|_{0,K} . \quad \square \end{aligned}$$

### Theorem 5.4 Consistency error estimate II

Let  $\hat{Q}_{\hat{K}}$  be a quadrature formula with respect to the reference element  $\hat{K}$  with positive weights  $\hat{\omega}_{\ell, \hat{K}}, 1 \leq \ell \leq L$ , and the property that there exists  $k \in \mathbb{N}$  such that

$$(5.42) \quad \hat{P}_{\hat{K}} = P_k(\hat{K}) ,$$

$$(5.43) \quad \hat{E}_{\hat{K}}(\hat{\varphi}) = 0 , \quad \hat{\varphi} \in P_{2k-2}(\hat{K}) .$$

Assume that  $g \in W^{k,p}(\Omega)$  and  $p \in P_k(K)$  and suppose that  $k \in \mathbb{N}$  satisfies

$$(5.44) \quad k > \frac{d}{p} .$$

Then, there exists a constant  $C \in \mathbb{R}_+$ , independent of  $h \in \mathcal{H}$ , such that for all  $K \in \mathcal{T}_h$

$$(5.45) \quad |E_K(gp)| \leq C (\text{meas}(K))^{(\frac{1}{2}-\frac{1}{p})} h_K^k \|g\|_{k,p,K} \|p\|_{1,K} .$$

**Proof.** In view of (5.18) in Lemma 5.1, we have

$$E_K(gp) = \det(B_K) \hat{E}_{\hat{K}}(\hat{g}\hat{p})$$

where

$$\hat{g} \in W^{k,p}(\hat{K}) \quad , \quad \hat{p} \in P_k(\hat{K}) .$$

Denoting by  $\hat{Q} : L^2(\hat{K}) \rightarrow P_1(\hat{K})$  the **orthogonal projection** of  $L^2(\hat{K})$  onto  $P_k(\hat{K})$ , we split  $\hat{E}_{\hat{K}}(\hat{g}\hat{p})$  according to

$$(5.46) \quad \hat{E}_{\hat{K}}(\hat{g}\hat{p}) = \hat{E}_{\hat{K}}(\hat{g}\hat{Q}\hat{p}) + \hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))$$

and estimate both parts separately. (Note that a direct estimation (without such a splitting) would result in a non optimal estimate).

**(i) Estimation of the first term in (5.46)**

In view of (5.44) and the **Sobolev imbedding theorem**, the space  $W^{k,p}(\hat{K})$  is continuously imbedded in  $C^0(\hat{K})$ . Hence, for  $\hat{\psi} \in W^{k,p}(\hat{K})$

$$|\hat{E}_{\hat{K}}(\hat{\psi})| \leq \hat{C} |\hat{\psi}|_{0,\infty,\hat{K}} \leq \hat{C} \|\hat{\psi}\|_{k,p,\hat{K}} .$$

Since

$$\hat{E}_{\hat{K}}(\hat{\psi}) = 0 \quad , \quad \hat{\psi} \in P_{k-1}(\hat{K}) ,$$

the **Bramble-Hilbert lemma** infers

$$(5.47) \quad |\hat{E}_{\hat{K}}(\hat{\psi})| \leq \hat{C} |\hat{\psi}|_{k,p,\hat{K}} .$$

Now, for  $\hat{\psi} = \hat{g}\hat{Q}\hat{p}$ , observing that

$$|\hat{Q}\hat{p}|_{\ell,\infty,\hat{K}} = 0 \quad , \quad \ell \geq 2 ,$$

the **generalized Leibniz formula** implies

$$|\hat{g}\hat{Q}\hat{p}|_{k,p,\hat{K}} \leq \hat{C} \left( |\hat{g}|_{k,p,\hat{K}} |\hat{Q}\hat{p}|_{0,\infty,\hat{K}} + |\hat{g}|_{k-1,p,\hat{K}} |\hat{Q}\hat{p}|_{1,\infty,\hat{K}} \right).$$

Taking advantage of the equivalence of norms on  $P_1(\hat{K})$ , we get

$$(5.48) \quad |\hat{g}\hat{Q}\hat{p}|_{k,p,\hat{K}} \leq \hat{C} \left( |\hat{g}|_{k,p,\hat{K}} |\hat{Q}\hat{p}|_{0,\hat{K}} + |\hat{g}|_{k-1,p,\hat{K}} |\hat{Q}\hat{p}|_{1,\hat{K}} \right).$$

Since  $\hat{Q}$  is an orthogonal projection, we have

$$(5.49) \quad |\hat{Q}\hat{p}|_{0,\hat{K}} \leq |\hat{p}|_{0,\hat{K}}.$$

Moreover, since  $\hat{Q}\hat{p} = \hat{p}$ ,  $\hat{p} \in P_0(\hat{K})$ , the **Bramble-Hilbert lemma** gives

$$|\hat{p} - \hat{Q}\hat{p}|_{1,\hat{K}} \leq \hat{C} |\hat{p}|_{1,\hat{K}},$$

whence

$$(5.50) \quad |\hat{Q}\hat{p}|_{1,\hat{K}} \leq |\hat{p} - \hat{Q}\hat{p}|_{1,\hat{K}} + |\hat{p}|_{1,\hat{K}} \leq (1 + \hat{C}) |\hat{p}|_{1,\hat{K}}.$$

Combining (5.47)-(5.50), we obtain

$$(5.51) \quad \hat{E}_{\hat{K}}(\hat{g}\hat{Q}\hat{p}) \leq \hat{C} \left( |\hat{g}|_{k,p,\hat{K}} |\hat{p}|_{0,\hat{K}} + |\hat{g}|_{k-1,p,\hat{K}} |\hat{p}|_{1,\hat{K}} \right).$$

**(ii) Estimation of the second term in (5.46)**

Since  $\hat{p} - \hat{Q}\hat{p} = 0$  for  $\hat{p} \in P_1(\hat{K})$ , we may assume  $k \geq 2$ .

We further choose  $r \in [1, \infty)$  large enough so that the mappings

$$W^{k,p}(\hat{K}) \rightarrow W^{k-1,r}(\hat{K}) \quad , \quad W^{k-1,r}(\hat{K}) \rightarrow C^0(\hat{K})$$

represent continuous imbeddings.

**Case 1**  $1 \leq p < d$

We may choose  $\frac{1}{r} = \frac{1}{p} - \frac{1}{d}$  so that by the **Sobolev imbedding theorem** the mapping  $W^{1,p}(\hat{K}) \rightarrow L^r(\hat{K})$  and hence, also the mapping  $W^{k,p}(\hat{K}) \rightarrow W^{k-1,r}(\hat{K})$  is a continuous imbedding.

Moreover, in view of (5.44), we have

$$k - 1 - \frac{d}{r} = k - \frac{d}{p} > 0.$$

Then, the **Sobolev imbedding theorem** also guarantees that the mapping  $W^{k-1,r}(\hat{K}) \rightarrow C^0(\hat{K})$  is a continuous imbedding.

**Case 2**  $d < p$

In this case, due to the **Sobolev imbedding theorem** the mapping  $W^{1,p}(\hat{K}) \rightarrow L^r(\hat{K})$  is a continuous imbedding for all  $r \in [1, \infty]$ , if  $d < p$ , and for all  $r \in [1, \infty)$ , if  $d = p$ . It follows that the same holds

true for the mapping  $W^{k,p}(\hat{K}) \rightarrow W^{k-1,r}(\hat{K})$ .  
Further, choosing  $r$  so large that

$$k - 1 - \frac{d}{r} > 0 ,$$

the **Sobolev imbedding theorem** implies that  $W^{k-1,r}(\hat{K}) \rightarrow C^0(\hat{K})$  is a continuous imbedding.

Now, in either case we obtain

$$\begin{aligned} |\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| &\leq \hat{C} |\hat{g}(\hat{p} - \hat{Q}\hat{p})|_{0,\infty,\hat{K}} \leq \\ &\leq \hat{C} |\hat{g}|_{0,\infty,\hat{K}} |\hat{p} - \hat{Q}\hat{p}|_{0,\infty,\hat{K}} \leq \\ &\leq \hat{C} \|\hat{g}\|_{k-1,r,\hat{K}} |\hat{p} - \hat{Q}\hat{p}|_{0,\infty,\hat{K}} . \end{aligned}$$

Consequently, the linear functional  $\hat{E}_{\hat{K}}(\hat{p}) : W^{k-1,r}(\hat{K}) \rightarrow \mathbb{R}$  given by  $(\hat{E}_{\hat{K}}(\hat{p}))(\hat{g}) := \hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))$  is continuous with

$$\|\hat{E}_{\hat{K}}(\hat{p})\| \leq \hat{C} \|\hat{p} - \hat{Q}\hat{p}\|_{0,\infty,\hat{K}} .$$

Since

$$\hat{E}_{\hat{K}}(\hat{p}) = 0 \quad , \quad \hat{p} \in P_{k-2}(\hat{K}) ,$$

by the **Bramble-Hilbert lemma**

$$(5.52) \quad |\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| \leq \hat{C} |\hat{g}|_{k-1,r,\hat{K}} |\hat{p} - \hat{Q}\hat{p}|_{0,\hat{K}} .$$

Moreover,  $\hat{Q}\hat{p} = \hat{p}$ ,  $\hat{p} \in P_0(\hat{K})$ , and hence,

$$(5.53) \quad |\hat{p} - \hat{Q}\hat{p}|_{0,\hat{K}} \leq \hat{C} |\hat{p}|_{1,\hat{K}} .$$

In order to estimate  $|\hat{g}|_{k-1,r,\hat{K}}$  in (5.52), we note that in view of the **Sobolev imbedding theorem** the mapping

$$W^{1,p}(\hat{k}) \rightarrow L^r(\hat{k})$$

is a continuous imbedding, whence

$$|\hat{f}|_{0,r,\hat{K}} \leq \hat{C} \left( |\hat{f}|_{0,p,\hat{K}} + |\hat{f}|_{1,p,\hat{K}} \right) ,$$

and hence,

$$(5.54) \quad |\hat{g}|_{k,r,\hat{K}} \leq \hat{C} \left( |\hat{g}|_{k-1,p,\hat{K}} + |\hat{g}|_{k,p,\hat{K}} \right) .$$

Using (5.53) and (5.54) in (5.52), we get

$$(5.55) \quad |\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| \leq \hat{C} \left( |\hat{g}|_{k-1,p,\hat{K}} + |\hat{g}|_{k,p,\hat{K}} \right) |\hat{p}|_{1,\hat{K}} .$$

Finally, combining (5.46),(5.51),(5.55) and observing that Theorem 4.3 infers

$$|\hat{g}|_{k-j,p,\hat{K}} \leq \hat{C} (\det(B_K))^{-1/p} h_K^{k-j} |g|_{k-j,p,K} \quad , \quad 0 \leq j \leq 1 ,$$

$$|\hat{p}|_{j,\hat{K}} \leq \hat{C} (\det(B_K))^{-1/2} h_K^j |p|_{j,K} \quad , \quad 0 \leq j \leq 1 \quad ,$$

we arrive at the assertion.  $\square$

After having estimated both the approximation and the consistency errors, we are now in a position to establish an **a priori estimate of the global discretization error** in the  $H^1(\Omega)$ -norm.

For this purpose, we provide the following auxiliary result.

**Lemma 5.3 Generalized Schwarz inequality for sums**

Assume  $a_i, b_i, c_i \in \mathbb{R}$ ,  $1 \leq i \leq m$ , and  $r_1, r_2, r_3 \in \mathbb{N}$  satisfying  $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ . Then, there holds

$$(5.56) \quad \sum_{i=1}^m |a_i b_i c_i| \leq \left( \sum_{i=1}^m |a_i|^{r_1} \right)^{1/r_1} \left( \sum_{i=1}^m |b_i|^{r_2} \right)^{1/r_2} \left( \sum_{i=1}^m |c_i|^{r_3} \right)^{1/r_3} .$$

**Proof.** The proof is left as an exercise.  $\square$

**Theorem 5.5 A priori estimate in the  $H^1(\Omega)$ -norm**

In addition to the assumptions **(A1),(A2),(A3)**, suppose that for some integer  $k \in \mathbb{N}$  we have

$$(5.57) \quad \hat{P} = P_k(\hat{K}) \quad ,$$

and that

$$(5.58) \quad H^{k+1}(\hat{K}) \rightarrow C^s(\hat{K})$$

is a continuous imbedding, where  $s \in \mathbb{N}_0$  is the maximal order of partial derivatives in the definition of the set  $\hat{\Sigma}_{\hat{K}}$  of degrees of freedom.

Moreover, let  $\hat{Q}_{\hat{K}}$  be a quadrature formula with respect to the reference element  $\hat{K}$  with positive weights  $\hat{\omega}_{\ell,\hat{K}}$ ,  $1 \leq \ell \leq L$ , satisfying

$$(5.59) \quad \hat{E}_{\hat{K}}(\hat{\varphi}) = 0 \quad , \quad \hat{\varphi} \in P_{2k-2}(\hat{K}) .$$

As far as the data of the model problem (5.9) are concerned, we require the existence of some  $p \geq 2$  with  $k > \frac{d}{p}$  such that the following **regularity assumptions** are met

$$(5.60) \quad a_{ij} \in W^{k,\infty}(\Omega) \quad , \quad 1 \leq i, j \leq d \quad , \quad f \in W^{k,p}(\Omega) .$$

We denote by  $u \in V := H_0^1(\Omega)$  the solution of (5.9) and by  $u_h \in V_h$ ,  $h \in \mathcal{H}$  the solution of its finite element approximation (5.21), and we assume

$$(5.61) \quad u \in V \cap H^{k+1}(\Omega) .$$

Then, there exists a positive constant  $C \in \mathbb{R}$ , independent of  $h \in \mathcal{H}$ , such that the following **a priori error estimate** for the global discretization error holds true

$$(5.62) \quad \|u - u_h\|_{1,\Omega} \leq C h^k \left( |u|_{k+1,\Omega} + \sum_{i,j=1}^d \|a_{ij}\|_{k,\infty,\Omega} \|u\|_{k+1,\Omega} + \|f\|_{k,p,\Omega} \right).$$

**Proof.** The assumptions apply that the family  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  of approximate bilinear forms  $a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$ ,  $h \in \mathcal{H}$ , is **uniformly  $V_h$ -elliptic**. We may thus apply **Strang's first lemma** and provide appropriate estimates of the **approximation error** and the **consistency error**.

**(i) Approximation error**

In view of the regularity assumption (5.61), it follows from Theorem 4.5 that

$$(5.63) \quad \|u - \Pi_h u\|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega}.$$

**(ii) Consistency error, Part I**

By Theorem 5.3 and the Cauchy-Schwarz inequality, for  $w_h \in V_h$  we obtain

$$(5.64) \quad \begin{aligned} |a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)| &\leq \\ &\leq \sum_{K \in \mathcal{T}_h} \sum_{i,j=1}^d |E_K(a_{ij} \frac{\partial \Pi_K u}{\partial x_i} \frac{\partial w_h}{\partial x_j})| \leq \\ &\leq C \sum_{K \in \mathcal{T}_h} h_K^k \left( \sum_{i,j=1}^d \|a_{ij}\|_{k,\infty,K} \right) \|\Pi_K u\|_{k,K} |w_h|_{1,K} \leq \\ &\leq C h^k \left( \sum_{i,j=1}^d \|a_{ij}\|_{k,\infty,K} \right) \left( \sum_{K \in \mathcal{T}_h} \|\Pi_K u\|_{k,K}^2 \right)^{1/2} |w_h|_{1,\Omega}. \end{aligned}$$

Applying Theorem 4.5 once more, we get

$$(5.65) \quad \begin{aligned} \left( \sum_{K \in \mathcal{T}_h} \|\Pi_K u\|_{k,K}^2 \right)^{1/2} &\leq \\ &\leq \|u\|_{k,K} + \left( \sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{k,K}^2 \right)^{1/2} \leq \\ &\leq \|u\|_{k,K} + C h |u|_{k+1,\Omega} \leq C \|u\|_{k+1,\Omega}. \end{aligned}$$

Using (5.65) in (5.64) yields

$$(5.66) \quad \sup_{w_h \in V_h} \frac{|a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)|}{\|w_h\|_{1,\Omega}} \leq$$
$$(5.67) \quad \leq C h^k \sum_{i,j=1}^d \|a_{ij}\|_{k,\infty,\Omega} \|u\|_{k+1,\Omega} .$$

### (ii) Consistency error, Part II

Using Theorem 5.4, for  $w_h \in V_h$  it follows that

$$(5.68) \quad \begin{aligned} |\ell(w_h) - \ell_h(w_h)| &\leq \sum_{K \in \mathcal{T}_h} |E_K(fw_h)| \leq \\ &\leq C \sum_{K \in \mathcal{T}_h} (\text{meas}(K))^{(\frac{1}{2} - \frac{1}{p})} h_K^k \|f\|_{k,p,K} \|w_h\|_{1,K} . \end{aligned}$$

Applying the **generalized Schwarz inequality** (Lemma 5.3) to the last sum in (5.68) with

$$m := \text{card}(\mathcal{T}_h) , \quad \frac{1}{r_1} := \frac{1}{2} - \frac{1}{p} , \quad r_2 := p , \quad r_3 := 2 ,$$

results in

$$(5.69) \quad \begin{aligned} |\ell(w_h) - \ell_h(w_h)| &\leq \sum_{K \in \mathcal{T}_h} |E_K(fw_h)| \leq \\ &\leq C h^k (\text{meas}(\Omega))^{(\frac{1}{2} - \frac{1}{p})} \|f\|_{k,p,\Omega} \|w_h\|_{1,\Omega} . \end{aligned}$$

We thus get

$$(5.70) \quad \sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_{1,\Omega}} \leq C h^k (\text{meas}(\Omega))^{(\frac{1}{2} - \frac{1}{p})} \|f\|_{k,p,\Omega} .$$

Finally, combining (5.63),(5.66) and (5.70), the assertion is a direct consequence of Strang's first lemma.  $\square$

## 5.2.2 Special quadrature formulas

We first consider **quadrature formulas** for **simplicial triangulations**.

### Lemma 5.4 Quadrature formulas for simplicial triangulations

Let  $\hat{K}$  be a non degenerate 2-simplex with vertices  $\hat{a}_i, 1 \leq i \leq 3$ , and denote by  $\hat{a}_{ij}, 1 \leq i < j \leq 3$ , the midpoints of the edges and by  $\hat{a}_{123}$  the center of gravity. then, there holds:

(i) The quadrature formula

$$(5.71) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) d\hat{x} \approx \text{meas}(\hat{K}) \hat{\varphi}(\hat{a}_{123})$$

is exact for polynomials  $\hat{p} \in P_1(\hat{K})$ , i.e.,

$$(5.72) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in P_1(\hat{K}) .$$



(ii) The quadrature formula

$$(5.73) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \frac{1}{3} \operatorname{meas}(\hat{K}) \sum_{1 \leq i < j \leq 3} \hat{\varphi}(\hat{a}_{ij})$$

is exact for polynomials  $\hat{p} \in P_2(\hat{K})$ , i.e.,

$$(5.74) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in P_2(\hat{K}) \, .$$

(iii) The quadrature formula

$$(5.75) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \frac{1}{60} \operatorname{meas}(\hat{K}) \left( 3 \sum_{i=1}^3 \hat{\varphi}(\hat{a}_i) + 8 \sum_{1 \leq i < j \leq 3} \hat{\varphi}(\hat{a}_{ij}) + 27 \hat{\varphi}(\hat{a}_{123}) \right)$$

is exact for polynomials  $\hat{p} \in P_3(\hat{K})$ , i.e.,

$$(5.76) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in P_3(\hat{K}) \, .$$

**Proof.** Let  $\lambda_i(\hat{x})$ ,  $1 \leq i \leq 3$ , be the barycentric coordinates of  $\hat{x} \in \hat{K}$ . Then, for  $\alpha_i \in \mathbb{N}_0$ ,  $1 \leq i \leq 3$ , there holds

$$(5.77) \quad \int_{\hat{K}} \prod_{i=1}^3 \lambda_i^{\alpha_i}(\hat{x}) \, d\hat{x} = \operatorname{meas}(\hat{K}) \frac{d! \prod_{i=1}^3 \alpha_i!}{\left( \sum_{i=1}^3 \alpha_i + d \right)!} \, .$$

The proof of (5.77) is left as an exercise.  $\square$

We next consider **quadrature formulas** for **rectangular triangulations**.

**Lemma 5.5** **Quadrature formulas for rectangular triangulations**

Let  $\hat{K} := [0, 1]^2$  with vertices

$$\hat{a}_1 := (0, 0) \, , \quad \hat{a}_2 := (1, 0) \, , \quad \hat{a}_3 := (1, 1) \, , \quad \hat{a}_4 := (0, 1)$$

and center  $\hat{a}_s := (1/2, 1/2)$ . Then, there holds (i) The quadrature formula

$$(5.78) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \frac{1}{4} \sum_{i=1}^4 \hat{\varphi}(\hat{a}_i)$$

is exact for polynomials  $\hat{p} \in Q_1(\hat{K})$ , i.e.,

$$(5.79) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in Q_1(\hat{K}) \, .$$

(ii) The quadrature formula

$$(5.80) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \hat{\varphi}(\hat{a}_s)$$

is exact for polynomials  $\hat{p} \in Q_1(\hat{K})$ , i.e.,

$$(5.81) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in Q_1(\hat{K}) \, .$$

(iii) Let  $k \in \mathbb{N}_0$  and denote by  $\hat{b}_i \in (0, 1)$ ,  $1 \leq i \leq k+1$ , the **nodes** and by  $\hat{\omega}_i$ ,  $1 \leq i \leq k+1$ , the **weights** of the **Gauss-Legendre quadrature formula**

$$\int_0^1 \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \sum_{i=1}^{k+1} \hat{\omega}_i \hat{\varphi}(\hat{b}_i) \, ,$$

which is exact for polynomials  $\hat{p} \in P_{2k+1}(\hat{k})$ .

Then, the quadrature formula

$$(5.82) \quad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \, d\hat{x} \approx \sum_{i_1, i_2 \in \{1, \dots, k+1\}} \hat{\omega}_{i_1} \hat{\omega}_{i_2} \hat{\varphi}(\hat{b}_{i_1}, \hat{b}_{i_2})$$

is exact for polynomials  $\hat{p} \in Q_{2k+1}(\hat{K})$ , i.e.,

$$(5.83) \quad \hat{E}(\hat{p}) = 0 \quad , \quad \hat{p} \in Q_{2k+1}(\hat{K}) \, .$$

**Proof.** The proof is left as an exercise. □

We note that the quadrature formulas (5.71),(5.73),(5.75) and (5.78), (5.80),(5.82) can be easily generalized to 3D.

### 5.3 Strang's second lemma

In this section, we will derive an **abstract error estimate**, known as **Strang's second lemma**, that is applicable for finite element approximations of variational equations

$$(5.84) \quad a(u, v) = \ell(v) \quad , \quad v \in V \, ,$$

where the finite element spaces  $V_h$ ,  $h \in \mathcal{H}$ , are no longer subspaces of the underlying infinite dimensional function space  $V$ .

We assume that  $(V_h)_{h \in \mathcal{H}}$  is a family of finite dimensional linear spaces and suppose that  $\|\cdot\|_h$ ,  $h \in \mathcal{T}_h$ , provides a **mesh-dependent norm** on  $V_h + V$ . We further suppose that the linear functional  $\ell$  is well defined on  $V_h + V$ .

We consider an associated family of **approximate bilinear forms**

$$(5.85) \quad a_h(\cdot, \cdot) : (V_h + V) \times (V_h + V) \rightarrow \mathbb{R} \, .$$

We suppose that the family  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  is **uniformly  $V_h$ -elliptic** in the sense that there exists a positive constant  $\tilde{\alpha}$ , independent of  $h \in \mathcal{H}$ , such that

$$(5.86) \quad a_h(v_h, v_h) \geq \tilde{\alpha} \|v_h\|_{V_h}^2, \quad v_h \in V_h, \quad h \in \mathcal{H}.$$

We further suppose that the family  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  is **uniformly bounded** on  $V_h + V$  in the sense that there exists a positive constant  $\tilde{M}$ , independent of  $h \in \mathcal{H}$ , such that

$$(5.87) \quad |a_h(u, v)| \leq \tilde{M} \|u\|_h \|v\|_h, \quad u, v \in V_h + V, \quad h \in \mathcal{H}.$$

We approximate (5.84) by the finite dimensional variational equations

$$(5.88) \quad a_h(u_h, v_h) = \ell(v_h), \quad v_h \in V_h, \quad h \in \mathcal{H}.$$

Strang's second lemma may be viewed as another generalization of Céa's lemma.

### Theorem 5.6 Strang's second lemma

Assume that  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  is a **uniformly bounded** and **uniformly  $V_h$ -elliptic** family of approximate bilinear forms and suppose that  $u \in V$  and  $u_h \in V_h, h \in \mathcal{H}$ , are the unique solutions of the variational equations (5.84) and (5.88), respectively.

Then, there exists a positive constant  $C \in \mathbb{R}$ , independent of  $h \in \mathcal{H}$ , such that

$$(5.89) \quad \|u - u_h\|_h \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - \ell(w_h)|}{\|w_h\|_h} \right).$$

**Proof.** Taking advantage of the **uniform  $V_h$ -ellipticity** and observing (5.88), for an arbitrary  $v_h \in V_h$  we obtain

$$\begin{aligned} \tilde{\alpha} \|u_h - v_h\|_h^2 &\leq a_h(u_h - v_h, u_h - v_h) = \\ &= a_h(u - v_h, u_h - v_h) + \left( \ell(u_h - v_h) - a_h(u, u_h - v_h) \right). \end{aligned}$$

By the **uniform boundedness** (5.87) we get

$$\begin{aligned} \tilde{\alpha} \|u_h - v_h\|_h &\leq a_h(u_h - v_h, u_h - v_h) \leq \\ &\leq \tilde{M} \|u - v_h\|_h + \frac{|\ell(u_h - v_h) - a_h(u, u_h - v_h)|}{\|u_h - v_h\|_h} \leq \\ &\leq \tilde{M} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|\ell(w_h) - a_h(u, w_h)|}{\|w_h\|_h}. \end{aligned}$$

In conjunction with the triangle inequality

$$\|u - u_h\|_h \leq \|u - v_h\|_h + \|u_h - v_h\|_h,$$

the previous inequality gives the assertion.  $\square$

As in the case of Strang's first lemma, we see that the upper bound for the global discretization error consists of an **approximation error**

$$(5.90) \quad \inf_{v_h \in V_h} \|u - v_h\|_h$$

and the **consistency error**

$$(5.91) \quad \sup_{w_h \in V_h} \frac{|\ell(w_h) - a_h(u, w_h)|}{\|w_h\|_h},$$

which can be estimated separately.

#### 5.4 A priori error estimate for nonconforming finite element approximations

As an example for a **nonconforming finite element** in the discretization of second order elliptic boundary value problems with respect to a family of shape-regular simplicial triangulations  $\mathcal{T}_h, h \in \mathcal{H}$ , of the computational domain  $\Omega \subset \mathbb{R}^d$  we consider the lowest order **Crouzeix-Raviart element**  $CR_1(K) := (K, P_K, \Sigma_K), K \in \mathcal{T}_h$ , also known as the **nonconforming P1-element**:

$$(5.92) \quad \begin{aligned} P_K &:= P_1(K), \\ \Sigma_K &:= \{ p(m_i(K)), 1 \leq i \leq d+1 \}, p \in P_K, \end{aligned}$$

where  $m_i(K), 1 \leq i \leq d+1$ , are the **midpoints** of the edges ( $d = 2$ ) and the **centers of gravity** of the faces ( $d = 3$ ), respectively (cf. Figure 5.1).

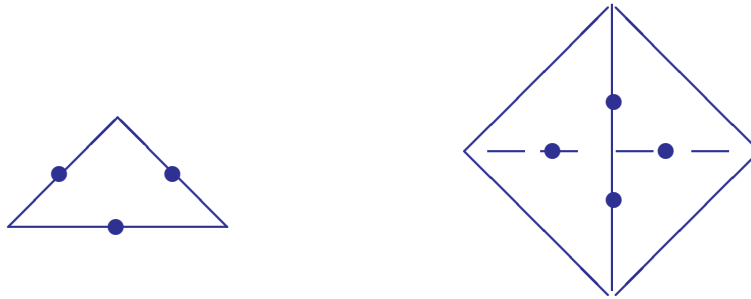


Fig. 5.1: Nonconforming Crouzeix-Raviart element

The associated **Crouzeix-Raviart finite element space**  $CR_1(\Omega, \mathcal{T}_h)$ , composed by the Crouzeix-Raviart elements  $CR_1(K), K \in \mathcal{T}_h$ , is then

given by

$$(5.93) \quad \begin{aligned} CR_1(\Omega, \mathcal{T}_h) &:= \\ &\{ v_h \in L^2(\Omega) \mid v_h|_K \in P_1(K), K \in \mathcal{T}_h, \\ &\quad v_h \text{ is continuous in } m(f), f \in \mathcal{F}_h(\Omega) \} , \end{aligned}$$

where  $m(f), f \in \mathcal{F}_h(\Omega)$  are the midpoints (centers of gravity) of interior edges (faces).

Obviously, in general

$$(5.94) \quad CR_1(\Omega, \mathcal{T}_h) \not\subset H^1(\Omega) .$$

The subspace  $CR_{1,0}(\Omega, \mathcal{T}_h)$  is defined by means of

$$CR_{1,0}(\Omega, \mathcal{T}_h) := \{ v_h \in CR_1(\Omega, \mathcal{T}_h) \mid v_h(m(f)) = 0, f \in \mathcal{F}_h(\partial\Omega) \} .$$

We consider the approximation of **Poisson's equation** under **homogeneous Dirichlet boundary conditions**

$$(5.95) \quad a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx = \ell(v) := \int_{\Omega} f v dx, v \in H_0^1(\Omega)$$

by

$$(5.96) \quad a_h(u_h, v_h) = \ell(v_h), \quad v_h \in CR_{1,0}(\Omega, \mathcal{T}_h), h \in \mathcal{H},$$

where the **mesh dependent bilinear form**

$$a_h(\cdot, \cdot) : H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) \times H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) \rightarrow \mathbb{R}$$

is given by

$$(5.97) \quad a_h(u, v) := \sum_{K \in \mathcal{T}_h(\Omega)} \int_K \nabla u \cdot \nabla v dx, u, v \in H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) .$$

We associate with (5.97) the **mesh dependent norm**

$$(5.98) \quad \|v\|_h := \sqrt{a_h(v, v)}, \quad v \in CR_{1,0}(\Omega, \mathcal{T}_h) \times H_0^1(\Omega) .$$

We will derive a **quasi-optimal a priori error estimate** in the  $\|\cdot\|_h$ -norm by an application of **Strang's second lemma**:

**Theorem 5.7 A priori error estimate in the  $\|\cdot\|_h$ -norm**

Let  $u \in H_0^1(\Omega) \cap H^s(\Omega)$ ,  $s = 2$  ( $d = 2$ ),  $s = 3$  ( $d = 3$ ), be the solution of (5.95) and  $u_h \in CR_{1,0}(\Omega, \mathcal{T}_h), h \in \mathcal{H}$ , its nonconforming P1-approximations. Then, there exists a constant  $C > 0$ , depending only on the shape regularity of the triangulations  $\mathcal{T}_h(\Omega), h \in \mathcal{H}$ , such that

$$(5.99) \quad \|u - u_h\|_h \leq C h |u|_{s,\Omega} .$$

**Proof.** It is easily verified that the family  $(a_h(\cdot, \cdot))_{h \in \mathcal{H}}$  of bilinear forms is **uniformly  $V_h$ -elliptic** ( $V_h := CR_{1,0}(\Omega, \mathcal{T}_h)$ ). According to **Strang's second lemma** we have to estimate the **approximation error**

$$(5.100) \quad \inf_{v_h \in CR_{1,0}(\Omega, \mathcal{T}_h)} \|u - v_h\|_h$$

and the **consistency error**

$$(5.101) \quad \sup_{w_h \in CR_{1,0}(\Omega, \mathcal{T}_h)} \frac{a_h(u, w_h) - \ell(w_h)}{\|w_h\|_h} .$$

**(i) Estimation of the approximation error**

Since  $S_{1,0}(\Omega, \mathcal{T}_h) \subset CR_{1,0}(\Omega, \mathcal{T}_h)$ , in view of Theorem 4.7 we have

$$(5.102) \quad \inf_{v_h \in CR_{1,0}(\Omega, \mathcal{T}_h)} \|u - v_h\|_h \leq \inf_{v_h \in S_{1,0}(\Omega, \mathcal{T}_h)} \|u - v_h\|_h \leq C h |u|_{s, \Omega} .$$

**(ii) Estimation of the consistency error**

Observing that

$$-\Delta u(x) = f(x) \quad \text{f.a.a. } x \in \Omega ,$$

by **Green's formula** we obtain for  $w_h \in CR_{1,0}(\Omega, \mathcal{T}_h)$

$$(5.103) \quad \begin{aligned} L_u(w_h) &:= a_h(u, w_h) - \ell(w_h) = \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K \nabla u \cdot \nabla w_h \, dx - \int_K f w_h \, dx \right) = \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K (-\Delta u - f) w_h \, dx + \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}} w_h \, d\sigma \right) = \\ &= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}_f} \int_f \frac{\partial u}{\partial \mathbf{n}} w_h \, d\sigma . \end{aligned}$$

Denoting by  $w_h|_f$  the **integral mean**

$$w_h|_f := \frac{1}{\text{meas}(f)} \int_f w_h \, d\sigma \quad , \quad f \in \mathcal{F}_h(\Omega) ,$$

it follows that

$$(5.104) \quad L_u(w_h) = \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}_f} \int_f \frac{\partial u}{\partial \mathbf{n}} (w_h - w_h|_f) \, d\sigma .$$

Moreover, since  $\frac{\partial I_h u}{\partial \mathbf{n}} \in P_0(K_\nu)$ ,  $1 \leq \nu \leq 2$ , we have

$$\int_{f \cap K_\nu} \frac{\partial I_h u}{\partial \mathbf{n}} (w_h - w_h|_f) d\sigma = 0 ,$$

and hence, (5.104) gives rise to

$$L_u(w_h) = \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}_f} \int_f \frac{\partial(u - I_h u)}{\partial \mathbf{n}} (w_h - w_h|_f) d\sigma .$$

The Cauchy-Schwarz inequality yields

$$|L_u(w_h)| \leq \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}_f} \left( \underbrace{\int_f \left| \frac{\partial(u - I_h u)}{\partial \mathbf{n}} \right|^2 d\sigma}_{:= I_1} \underbrace{\int_f |w_h - w_h|_f|^2 d\sigma}_{:= I_2} \right)^{1/2} .$$

We will estimate  $I_1$  and  $I_2$  separately.

**(ii)<sub>1</sub> Estimation of  $I_1$**

Taking advantage of the **affine equivalence** of the **Crouzeix-Raviart elements**, for the **reference element  $\hat{K}$**  we find by the **trace theorem** and the **Bramble-Hilbert lemma**

$$\int_{\partial \hat{K}} \left| \frac{\partial(u - I_h u)}{\partial \mathbf{n}} \right|^2 d\sigma \leq C \|u - I_h u\|_{2, \hat{K}}^2 \leq C |u|_{s, \hat{K}}^2 .$$

By a **standard scaling argument**

$$(5.105) \quad \int_{\partial K} \left| \frac{\partial(u - I_h u)}{\partial \mathbf{n}} \right|^2 d\sigma \leq C h |u|_{s, K}^2 .$$

**(ii)<sub>2</sub> Estimation of  $I_2$**

Again, for the **reference element  $\hat{K}$**  we find by the **Bramble-Hilbert lemma**

$$\int_{\hat{f}} |\hat{w}_h - \hat{w}_h|_{\hat{f}}|^2 d\hat{\sigma} \leq C |\hat{w}_h|_{1, \hat{K}}^2 , \quad \hat{w}_h \in P_1(\hat{K}) , \quad \hat{f} \in \mathcal{F}(\hat{K}) ,$$

and by a **standard scaling argument**, for  $w_h \in CR_{1,0}(\Omega, \mathcal{T}_h)$  and  $f \in \mathcal{F}(K)$ ,  $K \in \mathcal{T}_h$  we obtain

$$(5.106) \quad \int_f |w_h - w_h|_f|^2 d\sigma \leq C h |w_h|_{1, K}^2 .$$

Using (5.105) and (5.106), we finally obtain

$$\begin{aligned}
 |L_u(w_h)| &\leq 3 C h \sum_{K \in \mathcal{T}_h} |u|_{s,K} |w_h|_{1,K} \leq \\
 &\leq C h \left( \sum_{K \in \mathcal{T}_h} |u|_{s,K}^2 \sum_{K \in \mathcal{T}_h} |w_h|_{1,K}^2 \right)^{1/2} = C h |u|_{s,\Omega} \|w_h\|_h . \quad \square
 \end{aligned}$$