Chapter 6 A posteriori error estimates for finite element approximations

6.1 Introduction

The a posteriori error estimation of finite element approximations of elliptic boundary value problems has reached some state of maturity, as it is documented by a variety of monographs on this subject (cf., e.g., [1, 2, 3, 4, 5]). There are different concepts such as

- **residual type a posteriori error estimators,**
- **hierarchical type a posteriori error estimators,**
- **error estimators based on local averaging,**
- **error estimators based on the goal oriented weighted dual approach** (cf., in particular, [3]).

In this chapter, we will focus on **residual type a posteriori error estimators** and follow the exposition in [5].

We shall deal with the following **model problem**:
Let Ω be a bounded simply-connected polygonal domain in Euclidean space $\mathbb{R}^{\mathbf{2}}$ with boundary $\mathbf{\Gamma} = \mathbf{\Gamma_D} \cup \mathbf{\Gamma_N}$, $\mathbf{\Gamma_D} \cap \mathbf{\Gamma_N} = \emptyset$ and consider the elliptic boundary value problem

$$
\begin{aligned}
Lu \;:=\; -\operatorname{div}\,(a\;\mathbf{grad}\;u) \;&=\; f \quad \text{in } \Omega\ , \\
u \;&=\; 0 \quad \text{on } \Gamma_D\ , \\
\mathbf{n}\cdot a\;\mathbf{grad}\;u \;&=\; g \quad \text{on } \Gamma_N\ ,
\end{aligned}
\tag{6.1}
$$

where $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_N)$ and $a = (a_{ij})_{i,j=1}^2$ is supposed to be a matrix-valued function with entries $a_{ij} \in L^\infty(\Omega)$, that is symmetric, i.e.,

$$
a_{ij}(x) \;=\; a_{ji}(x) \quad \text{f.a.a. } x \in \Omega\ ,\ 1 \le i,j \le 2\ ,
$$

and uniformly positive definite in the sense that for almost all $x \in \Omega$

$$
\sum_{i,j=1}^{2} a_{ij}(x)\;\xi_i\;\xi_j \;\ge\; \alpha\;\mid \xi\mid^2 \quad , \quad \xi \in R^2\ ,\ \alpha > 0\ .
$$

The vector n denotes the exterior unit normal vector on Γ_N. We further set $\bar{\alpha} \;:=\; \max_{1\le i,j\le 2}\;\|a_{ij}\|_\infty$.
Setting

$$
H^1_{0,\Gamma_D}(\Omega) \;:=\; \{\; v \in H^1(\Omega)\;\mid\; v\,|_{\Gamma_D} = 0\;\}\ ,
$$

the **weak formulation** of (6.1) is as follows:
Find $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$
a(u,v) \;=\; \ell(v) \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega)\ ,
\tag{6.2}
$$

where

$$a(v,w) := \int_\Omega a \ \mathbf{grad}\ v \cdot \mathbf{grad}\ w \ dx \quad , \quad v,w \in H^1_{0,\Gamma_D}(\Omega) \ ,$$

$$\ell(v) := \int_\Omega f \ v \ dx \ + \int_{\Gamma_N} g \ v \ d\sigma \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega) \ .$$

Given a geometrically conforming simplicial triangulation $\mathcal{T}_h$ of Ω, we denote by

$$S_{1,\Gamma_D}(\Omega;\mathcal{T}_h) := \{ \ v_h \in H^1_{0,\Gamma_D}(\Omega) \ | \ v_h \,|_K \in P_1(K) \ , \ K \in \mathcal{T}_h \ \}$$

the trial space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_h$. Note that $P_k(K) \ , \ k \geq 0$, denotes the linear space of polynomials of degree $\leq k$ on K.
In the sequel we will refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D) \ , \ D \subseteq \bar{\Omega}$ as the sets of vertices and edges of $\mathcal{T}_h$ on D.
The **conforming P1 approximation** of (6.1) reads as follows:
Find $u_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$ such that

$$a(u_h,v_h) = \ell(v_h) \quad , \quad v_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h) \ . \tag{6.3}$$

Now, assuming that $\tilde{u}_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$ is some **iterative approximation** of $u_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$, we are interested in the **total error**

$$e := u - \tilde{u}_h \ ,$$

which is the sum of the **discretization error** $e_d := u - u_h$ and the **iteration error** $e_{it} := u_h - \tilde{u}_h$. It is easy to see that the total error e is in $V := H^1_{0,\Gamma_D}(\Omega)$ and satisfies the **error equation**

$$a(e,v) = r(v) \quad , \quad v \in V \ , \tag{6.4}$$

where $r(\cdot)$ stands for the residual with respect to the computed approximation $\tilde{u}_h$

$$r(v) := \int_\Omega f \ v \ dx \ + \int_{\Gamma_N} g \ v d\sigma \ - \ a(\tilde{u}_h,v) \quad , \quad v \in V \ . \tag{6.5}$$

We are interested in a cheaply computable **a posteriori error estimator** η of the total error e consisting of elementwise error contributions $\eta_K, K \in \mathcal{T}_h$ and edgewise error contributions $\eta_E, E \in \mathcal{E}_h$, in the sense that

$$\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2 \ + \sum_{E \in \mathcal{E}_h} \eta_E^2 \ , \tag{6.6}$$

which does provide **a lower and an upper bound** for e according to

$$\gamma \, \eta \ \leq \ \|e\|_{1,\Omega} \ \leq \ \Gamma \, \eta \tag{6.7}$$

with constants $0 < \gamma \leq \Gamma$ depending only on the ellipticity constants and on the shape regularity of the triangulation $\mathcal{T}_h$.

We may use the local error terms η_K and η_E as a criterion for **local refinements** of the elements $K \in \mathcal{T}_h$. Among several **refinement strategies**, the so-called **mean-value strategy** is as follows:
Compute the mean values

$$\bar{\eta}_K := \frac{1}{n_K} \sum_{K \in \mathcal{T}_h} \eta_K \ , \qquad (6.8)$$

$$\bar{\eta}_E := \frac{1}{n_E} \sum_{E \in \mathcal{E}_h} \eta_E \ ,$$

where $n_K := \text{card } \mathcal{T}_h$ and $n_E := \text{card } \mathcal{E}_h$.
Mark an element $K \in \mathcal{T}_h$ and an edge $E \in \mathcal{E}_h$ for refinement, if

$$\eta_K \ \geq \ \sigma \ \bar{\eta}_K \ , \qquad (6.9)$$

$$\eta_E \ \geq \ \sigma \ \bar{\eta}_E \ ,$$

where $0 < \sigma \leq 1$ is some appropriate **safety factor**, e.g., $\sigma = 0.9$.

Definition 6.1 Efficient and reliable a posteriori error estimators

An a posteriori error estimator η satisfying

$$\|e\|_{1,\Omega} \ \leq \ \Gamma \ \eta \qquad (6.10)$$

is called **reliable**, since it ensures a sufficient refinement in the sense that the H^1-norm of the total error e will be bounded by a quantity of the same order of magnitude as a user-prescribed accuracy, if this accuracy is tested by η.
On the other hand, an a posteriori error estimator η for which

$$\gamma \ \eta \ \leq \ \|e\|_{1,\Omega} \qquad (6.11)$$

is said to be **efficient**, since it underestimates the H^1-norm of the total error e and thus prevents too much refinement.

6.2 Residual based a posteriori error estimators

The residual based a posteriori error estimator can be derived by viewing the residual as an element of the dual space V^* and evaluating it with respect to the dual norm.

6.2.1 Upper bound for the total error

An important tool in the construction of an upper bound for the total error is **Clément's interpolation operator** which is defined as follows:

Definition 6.2 Clément's interpolation operator

For $p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)$ we denote by φ_p the basis function in $S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$ with supporting point p and we refer to D_p as the set

$$D_p \ := \ \bigcup \{ \ K \in \mathcal{T}_h \ \ | \ \ p \in \mathcal{N}_h(T) \ \} \ .0 \qquad (6.12)$$

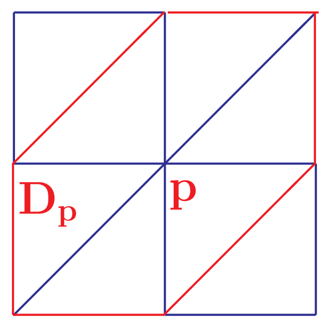

Fig. 6.1: Clément's interpolation operator (definition)

We refer to π_p as the L^2-projection onto $P_1(D_p)$, i.e.,

$$(\pi_p(v), w)_{0,D_p} \ = \ (v, w)_{0,D_p} \quad , \quad w \in P_1(D_p) \ ,$$

where $(\cdot,\cdot)_{0,D_p}$ stands for the L^2-inner product on $L^2(D_p) \times L^2(D_p)$. Then, **Clément's interpolation operator** P_C is defined as follows

$$P_C \ : \ L^2(\Omega) \longrightarrow S_{1,\Gamma_D}(\Omega, \mathcal{T}_h) \ , \qquad (6.13)$$

$$P_C v \ := \sum_{p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)} \pi_P(v) \ \varphi_P \ .$$

In order to establish **local approximation properties** of Clément's interpolation operator, for $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we introduce the sets

$$D_K^{(1)} \ := \ \bigcup \{ \ K' \in \mathcal{T}_h \ \ | \ \ \mathcal{N}_h(K') \cap \mathcal{N}_h(K) \ \neq \ \emptyset \ \} \ , \qquad (6.14)$$

$$D_E^{(1)} \ := \ \bigcup \{ \ K' \in \mathcal{T}_h \ \ | \ \ \mathcal{N}_h(E) \cap \mathcal{N}_h(K') \ \neq \ \emptyset \ \} \ . \qquad (6.15)$$

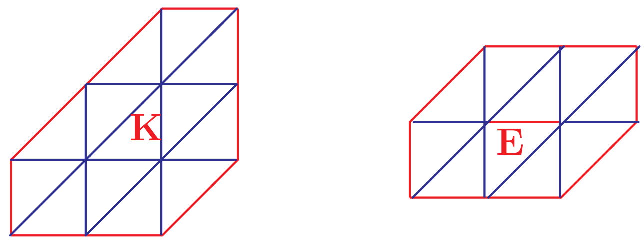

Fig. 6.2: Clément's interpolation operator (properties)

Using the **affine equivalence** of the elements and the **Bramble-Hilbert Lemma** one can show:

Theorem 6.1 Approximation properties of Clément's interpolation operator

For $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ let $D_K^{(1)}$ and $D_E^{(1)}$ be given by (6.14) and let P_C be Clément's interpolation operator as given by (6.13). Then, there exist constants $C_\nu > 0, 1 \le \nu \le 5$, depending only on the shape regularity of $\mathcal{T}_h$ such that for all $v \in S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$:

$$\|P_C\ v\|_{0,K} \ \le\ C_1\ \|v\|_{0,D_K^{(1)}}\ , \tag{6.16}$$

$$\|P_C\ v\|_{0,E} \ \le\ C_2\ \|v\|_{0,D_E^{(1)}}\ , \tag{6.17}$$

$$\|\mathbf{grad}\ P_C\ v\|_{0,K} \ \le\ C_3\ \|\mathbf{grad}\ v\|_{0,D_K^{(1)}}\ , \tag{6.18}$$

$$\|v\ -\ P_C\ v\|_{0,K} \ \le\ C_4\ h_K\ \|v\|_{1,D_K^{(1)}}\ , \tag{6.19}$$

$$\|v\ -\ P_C\ v\|_{0,E} \ \le\ C_5\ h_E^{1/2}\ \|v\|_{1,D_E^{(1)}}\ , \tag{6.20}$$

where $h_K := \text{diam } K$ and $h_E := \mid E \mid$.
Further, there exist constants $C_6, C_7 > 0$ depending only on the shape regularity of $\mathcal{T}_h$ such that

$$\Big(\sum_{K \in \mathcal{T}_h} \|v\|^2_{\mu,D_K^{(1)}}\Big)^{1/2} \ \le\ C_6\ \|v\|_{\mu,\Omega} \quad , \quad 0 \le \mu \le 1\ , \tag{6.21}$$

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} \|v\|^2_{\mu,D_E^{(1)}}\Big)^{1/2} \ \le\ C_7\ \|v\|_{\mu,\Omega} \quad , \quad 0 \le \mu \le 1\ . \tag{6.22}$$

Proof. We refer to [5]. □

We have now provided all prerequisites to establish an upper bound for the total error e measured in the H^1-norm. For functions $v_h \in W_0(\Omega; \mathcal{T}_h)$ we further refer to $[v_h]_J$ as the jump across the common edge $E \in \mathcal{E}_h(\Omega)$ of two adjacent elements $K_1, K_2 \in \mathcal{T}_h$

$$[v_h]_J \ := \ v_h\,|_{K_1} \ - v_h\,|_{K_2} \ .$$

Theorem 6.2 Upper bound for the total error
There exist constants $\Gamma_R, \Gamma_{osc} > 0$ and $\Gamma_{it} > 0$ depending only on the elipticity constants and the shape regularity of $\mathcal{T}_h$ such that

$$\|e\|_{1,\Omega} \ \leq \ \Gamma_R \ \eta_R \ + \ \Gamma_{osc} \ osc \ + \ \eta_{it} \ \|e_{it}\|_{1,\Omega} \ , \tag{6.23}$$

where the **element and edge residuals** are given by

$$\eta_R \ := \ \sum_{\nu=1}^{3} \eta_R^{(\nu)} \ ,$$

$$\eta_R^{(1)} \ := \ \Big(\sum_{K \in \mathcal{T}_h} h_T^2 \ \|\pi_h \ f - L\tilde{u}_h\|_{0,K}^2\Big)^{1/2} \ ,$$

$$\eta_R^{(2)} \ := \ \Big(\sum_{E \in \mathcal{E}_h(\Gamma_N)} h_E \ \|\pi_h g - \mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h\|_{0,E}^2\Big)^{1/2} \ ,$$

$$\eta_R^{(3)} \ := \ \Big(\sum_{E \in \mathcal{E}_h(\Omega)} h_E \ \|[\mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J\|_{0,E}^2\Big)^{1/2} \ ,$$

and osc stands for the **data oscillations**

$$osc \ := \ \Big(\sum_{K \in \mathcal{T}_h} osc_K^2 \ + \sum_{E \in \mathcal{E}_h(\Gamma_N)} osc_E^2\Big)^{1/2} \ ,$$

$$osc_K \ := \ h_T \ \|f - \pi_h f\|_{0,K} \quad , \quad osc_E \ := \ h_E \ \|g - \pi_h g\|_{0,E} \ .$$

Proof. Setting $v = e$ in (6.4), we obtain

$$\underline{\alpha} \ \|e\|_{1,\Omega}^2 \ \leq \ a(e,e) \ = \ r(e) \ = \ r(P_C e) \ + \ r(e - P_C e) \ . \tag{6.24}$$

Taking advantage of (6.3), for the first term on the right-hand side of (6.24) we get

$$r(P_C e) \ = \ \int_\Omega f \ P_C e \ dx \ + \ \int_{\Gamma_N} g \ P_C e \ d\sigma \ - \ a(\tilde{u}_h, P_C e) \ = \ = \ \sum_{K \in \mathcal{T}_h} a|_K(u_h - \tilde{u}_h, P_C e) \ .$$

Using (6.19), the Schwarz inequality, and observing (6.22), it follows that

$$
\begin{aligned}
(6.25)\qquad r(P_C e) \ &\le\ \overline{\alpha}\ C_3 \sum_{K\in\mathcal{T}_h} \|u_h - \tilde{u}_h\|_{1,K}\ \|e\|_{1,D_K^{(1)}} \ \le \\
&\le\ \overline{\alpha}\ C_3\ \Big(\sum_{K\in\mathcal{T}_h} \|u_h - \tilde{u}_h\|_{1,K}^2\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|e\|_{1,D_K^{(1)}}^2\Big)^{1/2} \ \le \\
&\le\ \overline{\alpha}\ C_3\ C_6\ \|e_{it}\|_{1,\Omega}\ \|e\|_{1,\Omega}\ .
\end{aligned}
$$

On the other hand, for the second term on the right-hand side of (6.24), Green's formula yields

$$
\begin{aligned}
r(e - P_C e) \ &=\ \int_\Omega f\ (e - P_C e)\ dx\ +\ \int_{\Gamma_N} g\ (e - P_C e)\ d\sigma\ + \\
&\quad +\ \sum_{K\in\mathcal{T}_h} \int_K \underbrace{\text{div}\ a\ \mathbf{grad}\ \tilde{u}_h}_{=\ -L\tilde{u}_h}\ (e - P_C e)\ dx\ - \\
&\quad -\ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbf{n}_{\partial K}\cdot a\ \mathbf{grad}\ \tilde{u}_h\ (e - P_C e)\ d\sigma\ = \\
&=\ \sum_{K\in\mathcal{T}_h} \int_K (\pi_h f - L\tilde{u}_h)\ (e - P_C e)\ dx\ + \\
&\quad +\ \sum_{E\in\mathcal{E}_h(\Omega)} \int_E [\mathbf{n}_E\cdot a\ \mathbf{grad}\ \tilde{u}_h]_J\ (e - P_C e)\ d\sigma\ + \\
&\quad +\ \sum_{E\in\mathcal{E}_h(\Gamma_N)} \int_E (\pi_h g - \mathbf{n}_E\cdot a\ \mathbf{grad}\ \tilde{u}_h)\ (e - P_C e)\ d\sigma\ + \\
&\quad +\ \sum_{K\in\mathcal{T}_h} \int_K (f - \pi_h f)\ (e - P_C e)\ dx\ + \\
&\quad +\ \sum_{E\in\mathcal{E}_h(\Gamma_N)} \int_E (g - \pi_h g)\ (e - P_C e)\ d\sigma\ .
\end{aligned}
$$

In view of (6.17),(6.18) and (6.22),(6.23), it follows that

$$
\begin{aligned}
(6.26)\qquad & r(e - P_C e)\ \le \\
&\le\ C_1\ C_6\ \Big(\sum_{K\in\mathcal{T}_h} h_K^2\ \|\pi_h f - L\tilde{u}_h\|_{0,K}^2\Big)^{1/2}\ \|e\|_{1,\Omega}\ + \\
&+\ C_2\ C_7\ \Big(\sum_{E\in\mathcal{E}_h(\Omega)} h_E\ \|[\mathbf{n}_E\cdot a\ \mathbf{grad}\ \tilde{u}_h]_J\|_{0,E}^2\Big)^{1/2}\ \|e\|_{1,\Omega}\ +
\end{aligned}
$$

$$+ \, C_2 \, C_7 \, (\sum_{E \in \mathcal{E}_h(\Gamma_N)} h_E \, \| \pi_h g - \mathbf{n}_E \cdot a \, \mathbf{grad} \, \tilde{u}_h \|_{0,E}^2)^{1/2} \, \|e\|_{1,\Omega} \, +$$

$$+ \, C_1 \, C_6 \, (\sum_{K \in \mathcal{T}_h} h_K^2 \, \| f - \pi_h f \|_{0,K}^2)^{1/2} \, \|e\|_{1,\Omega} \, +$$

$$+ \, C_2 \, C_7 \, (\sum_{E \in \mathcal{E}_h(\Gamma_N)} h_E \, \| g - \pi_h g \|_{0,E}^2)^{1/2} \, \|e\|_{1,\Omega} \, .$$

Using (6.25),(6.26) in (6.24), the assertion follows with $\Gamma_R = \Gamma_{osc} := \underline{\alpha}^{-1} \max(C_1 C_6, C_2 C_7)$ and $\Gamma_{it} := \underline{\alpha}^{-1} \, \overline{\alpha} \, C_3 C_6$. □

For the construction of a **lower bound** we will now show that the local contributions

$$\eta_{R,K}^{(\nu)} \; := \; \eta_R^{(\nu)} \, |_K \quad , \quad K \in \mathcal{T}_h \; , \; 1 \le \nu \le 3$$

of the residual based error estimator η_R do locally provide lower bounds for the total error e.

For this purpose we need appropriate localized polynomial functions defined on the elements K of the triangulation and the edges $E \in \mathcal{E}_h(\Omega) \cup E \in \mathcal{E}_h(\Gamma_N)$, respectively. Such functions are given by the **triangle-bubble functions** ψ_K and the **edge-bubble functions** ψ_E. In particular, denoting by $\lambda_i^K, 1 \le i \le 3$, the **barycentric coordinates** of $K \in \mathcal{T}_h$, then the triangle-bubble function ψ_K is defined by means of

$$\psi_K \; := \; 27 \, \lambda_1^K \, \lambda_2^K \, \lambda_3^K \; . \tag{6.27}$$

Note that $\text{supp} \, \psi_K = K_{int}$, i.e., $\psi_K \, |_{\partial K} = 0 \; , \; K \in \mathcal{T}_h$.

On the other hand, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ and $K \in \mathcal{T}_h$ such that $E \subset \partial K$ and $p_i^K \in \mathcal{N}_h(T) \; , \; 1 \le i \le 2$, we introduce the edge-bubble functions ψ_E according to

$$\psi_E \; := \; 4 \, \lambda_1^K \, \lambda_2^K \; . \tag{6.28}$$

Note that $\psi_E \, |_{E'} = 0$ for $E' \in \mathcal{E}_h(K), E' \ne E$.

The bubble functions ψ_K and ψ_E have the following important properties that can be easily verified taking advantage of the affine equivalence of the elements:

Lemma 6.1 Basic properties of the bubble functions. Part I

There exist constants $C_\nu > 0, 8 \le \nu \le 12$, depending only on the shape regularity of the triangulations $\mathcal{T}_h$ such that

$$\|p_h\|_{0,K}^2 \le C_8 \int_K p_h^2 \psi_K \, dx \quad , \quad p_h \in P_1(K) \, , \tag{6.29}$$

$$\|p_h\|_{0,E}^2 \le C_9 \int_E p_h^2 \psi_E \, d\sigma \quad , \quad p_h \in P_1(E) \, , \tag{6.30}$$

$$| \, p_h \, \psi_K \, |_{1,K} \le C_{10} \, h_K^{-1} \, \|p_h\|_{0,K} \quad , \quad p_h \in P_1(K) \, , \tag{6.31}$$

$$\|p_h \, \psi_K\|_{0,K} \le C_{11} \, \|p_h\|_{0,K} \quad , \quad p_h \in P_1(K) \, , \tag{6.32}$$

$$\|p_h \, \psi_E\|_{0,E} \le C_{12} \, \|p_h\|_{0,E} \quad , \quad p_h \in P_1(E) \, . \tag{6.33}$$

For functions $p_h \in P_1(E)$, $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we further need an **extension** $p_h^E \in L^2(K)$ where $K \in \mathcal{T}_h$ such that $E \subset \partial K$.
For this purpose we fix some $E' \subset \partial K$, $E' \neq E$, and for $x \in K$ denote by x_E that point on E such that $(x - x_E) \parallel E'$.
For $p_h \in P_1(E)$ we then set

$$p_h^E := p_h(x_E) \, . \tag{6.34}$$

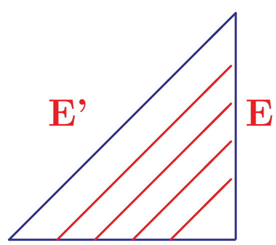

Fig. 6.3: Level lines of the extension p_h^E

Further, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we define $D_E^{(2)}$ as the union of elements $K \in \mathcal{T}_h$ containing E as a common edge

$$D_E^{(2)} := \bigcup \{ \, K \in \mathcal{T}_h \; | \; E \in \mathcal{E}_h(K) \, \} \, . \tag{6.35}$$

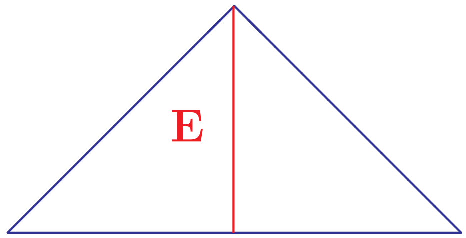

Fig. 6.4: The set $D_E^{(2)}$

We have the following properties of the extensions:

Lemma 6.2 Basic properties of the bubble functions. Part II

There exist constants $C_\nu > 0, 13 \le \nu \le 14$, depending only on the shape regularity of the triangulations $\mathcal{T}_h$ such that

$$(6.36) \quad |\, p_h^E\ \psi_E\,|_{1,D_E^{(2)}} \ \le\ C_{13}\ h_E^{-1/2}\ \|p_h\|_{0,e} \quad , \quad p_h \in P_1(E)\ ,$$

$$(6.37) \quad \|p_h^E\ \psi_E\|_{0,D_E^{(2)}} \ \le\ C_{14}\ h_E^{1/2}\ \|p_h\|_{0,E} \quad , \quad p_h \in P_1(E)\ .$$

Further, there exists a constant $C_{15} > 0$ independent of h_K, h_E such that for all $v \in V$ and $\mu = 0, 1$:

$$(6.38) \quad (\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} h_E^{1-\mu}\ \|v\|^2_{\mu,D_E^{(2)}})^{1/2} \ \le\ C_{15}\ (\sum_{K \in \mathcal{T}_h} h_K^{1-\mu}\ \|v\|^2_{\mu,K})^{1/2}\ .$$

We are now able to prove that up to higher order terms the estimator η_R also does provide a lower bound for the total error e:

Theorem 6.3 Lower bound for the total error

There exist constants $\gamma_R, \gamma_E > 0$, depending only on $\bar{\alpha}$ and on the shape regularity of $\mathcal{T}_h$ such that

$$(6.39) \qquad \gamma_R\ \eta_R\ -\ \gamma_E\ osc\ \le\ \|e\|_{1,\Omega}\ ,$$

where η_R and osc are given as in the previous theorem.

The theorem can be proved by a series of results which establish upper bounds for the local contributions $\eta_{R,T}^{(\nu)}, \le \nu \le 3$ of the estimator η_R.

Lemma 6.3 Upper bounds for the local contributions

(i) Let $K \in \mathcal{T}_h$. Then there holds:

$$(6.40) \qquad h_K\ \|\pi_h\ f - L\ \tilde{u}_h\|_{0,K}\ \le$$
$$\le\ \bar{\alpha}\ C_8\ C_{10}\ \|e\|_{1,K}\ +\ C_8\ C_{11}\ h_K\ \|f - \pi_h\ f\|_{0,K}\ .$$

(ii) Let $E \in \mathcal{E}_h(\Omega)$. Then there holds:

$$
\begin{aligned}
(6.41) \quad & h_E^{1/2} \|[\mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J\|_{0,E} \le \bar{\alpha} C_9 C_{13} \ \|e\|_{1,D_E^{(2)}} + \\
& + C_9 C_{14} h_E \|f - \pi_h \ f\|_{0,D_E^{(2)}} + C_9 C_{14} h_E \|\pi_h \ f - L\tilde{u}_h\|_{0,D_E^{(2)}} \ .
\end{aligned}
$$

(iii) Let $E \in \mathcal{E}_h(\Gamma_N)$. Then there holds:

$$
\begin{aligned}
(6.42) \quad & h_E^{1/2} \|\pi_h \ g - \mathbf{n}_E \cdot \mathbf{grad} \ \tilde{u}_h\|_{0,E} \le \\
& \bar{\alpha} C_9 C_{13} \|e\|_{1,D_E^{(2)}} + C_9 C_{12} h_E^{1/2} \|g - \ \pi_h \ g\|_{0,E} + \\
& + C_9 C_{14} h_E \|f - \pi_h \ f\|_{0,D_E^{(2)}} + C_9 C_{14} h_E \|\pi_h \ f - L\tilde{u}_h\|_{0,D_E^{(2)}} \ .
\end{aligned}
$$

Proof of (i) For the proof of (6.40) we set $p_h := \pi_h f$. Observing $\psi_K|_{\partial K} = 0$, by **Green's formula**

$$
\begin{aligned}
(6.43) \quad a|_K(\tilde{u}_h, p_h \ \psi_K) \ = \ & - \int_K \mathrm{div} \ (a \ \mathbf{grad} \ \tilde{u}_h) \ p_h \ \psi_K \ dx \ + \\
& + \int_{\partial K} \underbrace{\mathbf{n}_{\partial K} \cdot a \ \mathbf{grad} \ \tilde{u}_h \ p_h \ \psi_K}_{= \ 0} \ d\sigma \ .
\end{aligned}
$$

Then, using (6.30),(6.32) and (6.33) and taking advantage of (6.4),(6.43), it follows that

$$
\begin{aligned}
& \|\pi_h f - L \ \tilde{u}_h\|_{0,K}^2 \ \le \ C_8 \int_K (\pi_h f - L \ \tilde{u}_h) \ \pi_h \ \psi_K \ dx \ = \\
& = \ C_8 \ \Big(\int_K f \ \pi_h \ \psi_K \ dx \ - \ a|_K(\tilde{u}_h, \pi_h \ \psi_K) \ + \\
& + \int_K (\pi_h f - f) \ \pi_h \ \psi_K \ dx \Big) \ = \\
& = \ C_8 \ \Big(a|_K(e, \pi_h \ \psi_K) \ + \int_K (\pi_h f - f) \ \pi_h \ \psi_K \ dx \Big) \ \le \\
& \le \ C_8 \ C_{10} \ \overline{\alpha} \ h_K^{-1} \ \|e\|_{1,K} \ \|p_h\|_{0,K} \ + \ C_8 \ C_{11} \ \|\pi_h f - f\|_{0,K} \ \|p_h\|_{0,K} \ ,
\end{aligned}
$$

from which (6.40) can be easily deduced.

Proof of (ii) We set $p_h^E := [\mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J$. Again, in view of $\psi_E|_{E'} = 0, E' \neq E$,, Green's formula gives

$$\int_{\partial D_E^{(2)}} \mathbf{n}_{\partial D_E^{(2)}} \cdot a \ \mathbf{grad} \ \tilde{u}_h \ p_h^E \ \psi_E \ d\sigma \ = \tag{6.44}$$

$$= \ a|_{D_E^{(2)}}(\tilde{u}_h, p_h^E \ \psi_E) \ + \int_{D_E^{(2)}} \underbrace{\mathrm{div} \ a \ \mathbf{grad} \ \tilde{u}_h}_{= \ -L\tilde{u}_h} \ p_h^E \ \psi_E \ dx \ .$$

If we use (6.31),(6.37) and (6.38) and observe (6.4) and (6.44), it follows that

$$\|[\mathbf{n}_{\partial D_E^{(2)}} \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J\|_{0,E}^2 \ \le$$

$$\le \ C_9 \int_E [\mathbf{n}_{\partial D_E^{(2)}} \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J \ p_h^E \ \psi_E \ d\sigma \ =$$

$$= \int_{\partial D_E^{(2)}} [\mathbf{n}_{\partial D_E^{(2)}} \cdot a \ \mathbf{grad} \ \tilde{u}_h]_J \ p_h^E \ \psi_E \ d\sigma \ =$$

$$= \ C_9 \ \Big(a|_{D_E^{(2)}}(\tilde{u}_h, p_h^E \ \psi_E) \ - \int_{D_E^{(2)}} f \ p_h^E \ \psi_E dx \ +$$

$$+ \int_{D_E^{(2)}} (f - \pi_h f) \ p_h^E \ \psi_E dx \ + \int_{D_E^{(2)}} (\pi_h f - L\tilde{u}_h) \ p_h^E \ \psi_E dx \Big) \ =$$

$$= \ - \ C_9 \ a|_{D_E^{(2)}}(e, p_h^E \ \psi_E) \ +$$

$$+ \ C_9 \Big(\int_{D_E^{(2)}} (f - \pi_h f) \ p_h^E \ \psi_E dx \ + \int_{D_E^{(2)}} (\pi_h f - L\tilde{u}_h) \ p_h^E \ \psi_E dx \Big) \ \le$$

$$\le \ C_9 \ C_{13} \ \overline{\alpha} \ h_E^{-1/2} \ \|e\|_{1,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ +$$

$$+ \ C_9 \ C_{14} \ h_E^{1/2} \ \|f - \pi_h f\|_{0,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ +$$

$$+ \ C_9 \ C_{14} \ h_E^{1/2} \ \|\pi_h - L\tilde{u}_h\|_{0,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ ,$$

from which we readily deduce (6.41).

Proof of (iii) We set $p_h^E := \pi_h g - \mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h$. Observing $\psi_E|_{E'} = 0, E' \neq E$, by Green's formula we obtain

$$
\begin{aligned}
(6.45) \qquad & \int_E \mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h \ p_h^E \ \psi_E \ d\sigma \ = \\
& = \int_{\partial D_E^{(2)}} \mathbf{n}_{\partial D_E^{(2)}} \cdot a \ \mathbf{grad} \ \tilde{u}_h \ p_h^E \ \psi_E \ d\sigma \ = \\
& = \ a|_{D_E^{(2)}}(\tilde{u}_h, p_h^E \ \psi_E) \ + \int_{D_E^{(2)}} \underbrace{\operatorname{div} \ a \ \mathbf{grad} \ \tilde{u}_h}_{= \ -L\tilde{u}_h} \ p_h^E \ \psi_E \ dx \ .
\end{aligned}
$$

Now, using (6.31),(6.34),(6.37) and (6.38) and taking advantage of (6.4),(6.45), we get

$$
\begin{aligned}
& \|\pi_h g - \mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h\|_{0,E}^2 \ \leq \\
& \leq \ C_9 \int_E (\pi_h g - \mathbf{n}_E \cdot a \ \mathbf{grad} \ \tilde{u}_h) \ p_h^E \ \psi_E \ d\sigma \ = \\
& = \ C_9 \ \Big(\int_{D_E^{(2)}} f \ p_h^E \ \psi_E \ dx \ + \int_E g \ p_h^E \ \psi_E \ d\sigma \ - \ a|_{D_E^{(2)}}(\tilde{u}_h, p_h^E \ \psi_E) \ + \\
& + \int_{D_E^{(2)}} (\pi_h f - f) \ p_h^E \ \psi_E \ dx \ + \int_E (\pi_h g - g) \ f \ p_h^E \ \psi_E \ d\sigma \ - \\
& - \int_{D_E^{(2)}} (\pi_h f - L\tilde{u}_h) \ p_h^E \ \psi_E \ dx \Big) \ = \\
& = C_9 \ a|_{D_E^{(2)}}(e, p_h^E \ \psi_E) \ + \ C_9 \ \Big(\int_{D_E^{(2)}} (\pi_h f - f) \ p_h^E \ \psi_E \ dx \ + \\
& + \int_E (\pi_h g - g) \ f \ p_h^E \ \psi_E \ d\sigma \ - \int_{D_E^{(2)}} (\pi_h f - L\tilde{u}_h) \ p_h^E \ \psi_E \ dx \Big) \ \leq \\
& \leq \ C_9 \ C_{13} \ \overline{\alpha} \ h_E^{-1/2} \ \|e\|_{1,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ + \\
& + C_9 \ C_{14} \ h_E^{1/2} \ \|\pi_h f - f\|_{0,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ + \\
& + C_9 \ C_{12} \ \|\pi_h g - g\|_{0,E} \ \|p_h^E\|_{0,E} \ + \\
& + C_9 \ C_{14} \ h_E^{1/2} \ \|\pi_h f - L\tilde{u}_h\|_{0,D_E^{(2)}} \ \|p_h^E\|_{0,E} \ ,
\end{aligned}
$$

from which (6.41) follows easily. □

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