Chapter 7 Mixed finite element methods

7.1 The membrane problem revisited

We recall from Chapter 1 that the computation of the equilibrium state of a clamped membrane amounts to the solution of the **convex minimization problem**

$$J(u) \;=\; \inf_{v \in H_0^1(\Omega)} J(v) \; , \tag{7.1}$$

where $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ stands for the convex functional

$$J(v) \;:=\; \frac{1}{2} \int_\Omega |a \; \mathbf{grad} \; v|^2 \; dx \;-\; \int_\Omega f \; v \; dx \quad , \quad v \in H_0^1(\Omega) \; . \tag{7.2}$$

As we observed in Chapter 1, (7.1) is a particular example of a more general convex optimization problem of the form:
Given a Hilbert space V and a convex functional $J : V \rightarrow \mathbb{R}$, find $u \in V$ such that

$$J(u) \;=\; \inf_{v \in V} J(v) \; . \tag{7.3}$$

The optimization problem (7.3) can be given a **dual formulation** by means of the **Fenchel conjugate** of the functional J.

Definition 7.1 Fenchel conjugate

Let V be a Hilbert space with the dual space V^* and assume that $J : V \rightarrow \mathbb{R}$ is a convex functional. Then, the **Fenchel conjugate** $J^* : V^* \rightarrow \mathbb{R}$ of J is given by

$$J^*(v^*) \;:=\; \sup_{v \in V} \Big(< v^*, v > \;-\; J(v) \Big) \; , \tag{7.4}$$

where $< \cdot, \cdot >$ denotes the dual pairing between V and V^*.

Remark 7.1 The Fenchel conjugate in case $V = \mathbb{R}$

If $V = \mathbb{R}$, then $J^*(v^*)$ is the intercept with the v-axis of the tangent to J of slope v^*.

Theorem 7.1 Characterization of the Fenchel conjugate

Let V be a Hilbert space with the dual space V^* and assume that $J : V \rightarrow \mathbb{R}$ is a convex functional with $J^* : V^* \rightarrow \mathbb{R}$ denoting its Fenchel conjugate. Then, there holds

$$J(v) \;:=\; \sup_{v^* \in V^*} \Big(< v^*, v > \;-\; J^*(v^*) \Big) \; . \tag{7.5}$$

Proof. We refer to [3]. □

In view of (7.5), the optimization problem (7.3) can be written as the **saddle point problem**

$$\inf_{v\in V} \sup_{v^*\in V^*} \Big(< v^*, v > \; - \; J^*(v^*)\Big) \; . \tag{7.6}$$

Interchanging inf and sup, we obtain the **dual problem** to (7.3).

Definition 7.2 Dual problem

Let V be a Hilbert space with the dual space V^* and assume that $J : V \to \mathbb{R}$ is a convex functional with $J^* : V^* \to \mathbb{R}$ denoting its Fenchel conjugate. Then, the **dual problem** of (7.3) is given by

$$\sup_{v^*\in V^*} \inf_{v\in V} \Big(< v^*, v > \; - \; J^*(v^*)\Big) \; . \tag{7.7}$$

Remark 7.3 Characterization of the dual problem

Under some regularity assumptions (cf., e.g., [3]), there is no **duality gap**, i.e., inf and sup can be interchanged, and the dual problem (7.7) and the saddle point problem (7.6) are equivalent.

Now, focusing again our attention to the membrane problem (7.1), denoting by $\mathbf{H}(div,\Omega)$ the Hilbert space of square-integrable vector fields $\mathbf{q} \in L^2(\Omega)^2$ such that $\operatorname{div} \mathbf{q}$ lives in $L^2(\Omega)$ (cf. Section 7.2 below), we have that the **dual formulation** of (7.1) is given by

$$\inf_{\mathbf{q}\in\mathbf{H}(div,\Omega)} \sup_{v\in L^2(\Omega)} \Big(\frac{1}{2}\int_\Omega a^{-1}\, |\mathbf{q}|^2\, dx + \int_\Omega f\, v\, dx + \int_\Omega v\, \operatorname{div}\mathbf{q}\, dx\Big) \; . \tag{7.8}$$

The **necessary optimality conditions** give rise to the **saddle point problem**:

Find $(\mathbf{p}, u) \in \mathbf{H}(div,\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega a^{-1}\, \mathbf{p}\cdot\mathbf{q}\, dx + \int_\Omega u\, \operatorname{div}\, \mathbf{q}\, dx &= 0 \quad , \quad \mathbf{q} \in \mathbf{H}(div,\Omega) \; , \\ \int_\Omega \operatorname{div}\, \mathbf{p}\, v\, dx \qquad\qquad &= - \int_\Omega f\, v\, dx \quad , \quad v \in L^2(\Omega) \; . \end{aligned} \tag{7.9}$$

7.2 The Hilbert space $\mathbf{H}(div,\Omega)$

The dual problem associated with the membrane problem (7.1) involves the Hilbert space $\mathbf{H}(div,\Omega)$, which is defined as follows:

Definition 7.3 The Hilbert space $\mathbf{H}(div,\Omega)$

Let $\omega \subset \mathbb{R}^d$ be a Lipschitz domain. The space $H(\text{div};\Omega)$ is defined by

$$H(\text{div};\Omega) \; := \; \{ \; \mathbf{q} \in L^2(\Omega)^3 \mid \operatorname{div}\, \mathbf{q} \in L^2(\Omega) \; \} \; . \tag{7.10}$$

It is a Hilbert space with respect to the inner product

$$(7.11)\quad (\mathbf{j},\mathbf{q})_{div,\Omega} \ := \ (\mathbf{j},\mathbf{q})_{0,\Omega} + (\mathrm{div}\,\mathbf{j},\mathrm{div}\,\mathbf{q})_{0,\Omega} \quad , \quad \mathbf{j},\mathbf{q} \in H(\mathrm{div};\Omega)\ .$$

The associated norm will be denoted by $\|\cdot\|_{div,\Omega}$.

We set $\mathcal{D}(\bar{\Omega}) \ := \ \{\varphi|_\Omega \mid \varphi \in \mathcal{D}(\mathbb{R}^d)\}$. For vector fields $\mathbf{q} \in \mathcal{D}(\bar{\Omega})^d$ we consider the **normal component trace mapping**

$$(7.12)\qquad \eta_{\mathbf{n}}(\mathbf{q}) \quad := \quad \mathbf{n}\cdot\mathbf{q}|_\Gamma\ .$$

We recall that the space $\mathcal{D}(\bar{\Omega})^d$ is dense in $\mathbf{H}(\mathrm{div},\Omega)$ (cf., e.g., [2]; Thm. 2.1) and show:

Theorem 7.2 The normal component trace mapping
The normal component trace mapping $\eta_{\mathbf{n}}$ can be extended by continuity to a continuous linear mapping

$$(7.13)\qquad \eta_{\mathbf{n}}\ :\ H(\mathrm{div};\Omega)\ \rightarrow\ H^{-1/2}(\Gamma)\ .$$

Proof. (cf. [2]; Thm. 2.2): Let $\mathbf{q} \in \mathcal{D}(\bar{\Omega})^d$ and $\varphi \in \mathcal{D}(\bar{\Omega})$. By Green's formula

$$\int_\Omega \mathbf{q}\cdot\mathbf{grad}\ \varphi\ d\mathbf{x}\ +\ \int_\Omega \mathrm{div}\ \mathbf{q}\ \varphi d\mathbf{x}\ =\ \int_\Gamma \mathbf{n}\cdot\mathbf{q}\ \varphi\ d\sigma\ .$$

Since $\mathcal{D}(\bar{\Omega})$ is dense in $H^1(\Omega)$, the equation remains valid for $\varphi \in H^1(\Omega)$, whence

$$(7.14)\quad |\int_\Gamma \mathbf{n}\cdot\mathbf{q}\,\varphi\ d\sigma|\ \le\ \|\mathbf{q}\|_{div,\Omega}\ \|\varphi\|_{1,\Omega}\quad , \quad \mathbf{q}\in\mathcal{D}(\bar{\Omega})^d\ ,\ \varphi\in H^1(\Omega)\ .$$

Now, for $\mu \in H^{1/2}(\Gamma)$ there exists $\varphi \in H^1(\Omega)$ such that $\varphi|_\Gamma = \mu$ and $\|\varphi\|_{1,\Omega}\ \le\ \|\mu\|_{1/2,\Gamma}$. Then, (7.14) implies

$$|\int_\Gamma \mathbf{n}\cdot\mathbf{q}\ \varphi\ d\sigma|\ \le\ \|\mathbf{q}\|_{div,\Omega}\ \|\varphi\|_{1/2,\Gamma}\quad , \quad \mathbf{q}\in\mathcal{D}(\bar{\Omega})^d\ ,\ \mu\in H^{1/2}(\Gamma)\ .$$

Hence,

$$\|\eta_{\mathbf{n}}(\mathbf{q})\|_{-1/2,\Gamma}\ =\ \sup_{\mu\in H^{1/2}(\Gamma)} \frac{|\int_\Gamma \mathbf{n}\cdot\mathbf{q}\ \varphi\ d\sigma|}{\|\varphi\|_{1/2,\Gamma}}\ \le\ \|\mathbf{q}\|_{div,\Omega}$$

which proves the continuity of $\eta_{\mathbf{n}}$ with respect to the $\|\cdot\|_{div,\Omega}$-norm. Since $\mathcal{D}(\bar{\Omega})^d$ is dense in $\mathbf{H}(\mathrm{div},\Omega)$, $\eta_{\mathbf{n}}$ can be extended by continuity to $\mathbf{H}(\mathrm{div},\Omega)$ such that

$$(7.15)\qquad \|\eta_{\mathbf{n}}(\mathbf{q})\|_{-1/2,\Gamma}\ \le\ \|\mathbf{q}\|_{div,\Omega}\quad , \quad \mathbf{q}\in\mathbf{H}(\mathrm{div},\Omega)\ .$$

Denoting by $< \cdot, \cdot >_\Gamma$ the dual pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, **Green's formula** now reads as follows:

Theorem 7.3 Green's formula

For $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\varphi \in H^1(\Omega)$ there holds

$$\int_\Omega \mathbf{q} \cdot \mathbf{grad}\ \varphi\ d\mathbf{x} \ + \ \int_\Omega \mathrm{div}\ \mathbf{q}\ \varphi d\mathbf{x} \ = < \mathbf{n} \cdot \mathbf{q}, \varphi >_\Gamma \ . \tag{7.16}$$

Defintion 7.3 The subspace $\mathbf{H}_0(\mathrm{div}; \Omega)$

We define $\mathbf{H}_0(\mathrm{div}, \Omega)$ as the subspace of vector fields with vanishing normal components on Γ:

$$\mathbf{H}_0(\mathrm{div}, \Omega) \ := \ \{\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega) \mid \eta_\mathbf{n}(\mathbf{q}) = \mathbf{n} \cdot \mathbf{q}|_\Gamma = 0\} \ . \tag{7.17}$$

Remark 7.3 Characterization of $\mathbf{H}_0(\mathrm{div}, \Omega)$

Note that $\mathbf{H}_0(\mathrm{div}, \Omega)$ is the closure of $\mathcal{D}(\Omega)^d$ with respect to the $\|\cdot\|_{div,\Omega}$-norm.

We have $\mathrm{Ker}\ \eta_\mathbf{n} = \mathbf{H}_0(\mathrm{div}, \Omega)$. On the other hand, we can show that $\eta_\mathbf{n}$ is surjective.

Lemma 7.4 The range of the normal component trace mapping

The normal component mapping $\eta_\mathbf{n}$ is surjective, i.e., $\mathrm{Im}\ \eta_\mathbf{n} = H^{-1/2}(\Gamma)$.

Proof. For given $\mu^* \in H^{-1/2}(\Gamma)$ the **inhomogeneous Neumann problem**

$$\begin{aligned} -\ \Delta\varphi \ + \ \varphi \ &= \ 0 \quad \text{in } \Omega \ , \\ \mathbf{n} \cdot \mathbf{grad}\ \varphi \ &= \ \mu^* \quad \text{on } \Gamma \end{aligned}$$

has a unique weak solution in $H^1(\Omega)$.
Setting $\mathbf{q} = \mathbf{grad}\ \varphi$, we have $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\eta_\mathbf{n}(\mathbf{q}) = \mu^*$. □

Lemma 7.5 Properties of the normal component trace mapping

The mapping $\eta_\mathbf{n}$ is an **isometry** from $\mathbf{H}(\mathrm{div}, \Omega)$ onto $H^{-1/2}(\Gamma)$ i.e.,

$$\|\eta_\mathbf{n}\|_{\mathcal{L}(\mathbf{H}(div,\Omega);H^{-1/2}(\Gamma))} \ = \ 1 \ .$$

Proof. The proof is left as an exercise. □

7.3 Abstract saddle point problems

Let V and Q be Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$ and associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$ and assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$

and $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ are continuous bilinear forms. We denote by V^* and Q^* the dual spaces and further suppose that bounded linear functionals $f \in V^*$ and $g \in Q^*$ are given.
In this section, we investigate the **existence and uniqueness** of a solution to the **saddle point problem**:
Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) \; + \; b(v,p) \; &= \; < f, v >_{V^*,V} \quad , \quad v \in V \; , \\ b(u,q) \; &= \; < g, q >_{Q^*,Q} \quad , \quad q \in Q \; . \end{aligned} \tag{7.18}$$

Denoting by $A : V \to V^*$ and $B : V \to Q^*$ the bounded linear operators associated with the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ according to

$$< Au, v >_{V^*,V} \; = \; a(u,v) \quad , \quad u,v \in V \; ,$$

$$< Bv, q >_{Q^*,q} \; = \; a(u,v) \quad , \quad v \in V \; , \; q \in Q \; ,$$

the saddle point problem (7.18) can be equivalently written as the following **system of operator equations**

$$\begin{aligned} Au \; + \; B^*p \; &= \; f \quad \text{in } V^* \; , \\ Bu \; &= \; g \quad \text{in } Q^* \; . \end{aligned} \tag{7.19}$$

Let us first discuss an intuitive way how to solve (7.19):
If $g \in \mathrm{Im}(B)$, there exists $u_g \in V$ such that $Bu_g = g$. Then, setting $u = u_0 + u_g \; , \; u_0 \in \mathrm{Ker}(B)$, the first equation in (7.19) reads as

$$Au_0 \; + \; B^*p \; = \; f \; - \; Au_g \quad \text{in } V^* \; ,$$

Observing

$$< B^*p, v_0 >_{V^*,V} = < p, Bv_0 >_{Q,Q^*} \; = \; 0 \quad , \quad v_0 \in \mathrm{Ker}(B) \; ,$$

in variational form we obtain

$$a(u_0,v_0) \; = < f, v_0 > \; - \; a(u_g,v_0) \quad , \quad v_0 \in \mathrm{Ker}(B) \; . \tag{7.20}$$

If $\mathrm{Ker}(B) \subset V$ is closed and $a(\cdot,\cdot)$ is $\mathrm{Ker}(B)$**-elliptic**, then the **Lax-Milgram lemma** asserts the unique solvability of (7.20).
Once we know $u = u_0 + u_g$, we have to determine $p \in Q$ according to

$$B^*p \; = \; f \; - \; Au \; . \tag{7.21}$$

From (7.20) we know that

$$f \; - \; Au \; \in (\mathrm{Ker}(B))^0 \; .$$

where

$$(\mathrm{Ker}(B))^0 \; = \; \{v^* \in V^* \mid |v^*, v >_{V^*,V} \; = \; 0 \; , \; v \in \mathrm{Ker}(B)\} \; .$$

Hence, if

$$(\mathrm{Ker}(B))^0 \; = \; \mathrm{Im}(B^*) \; , \tag{7.22}$$

(7.21) admits a solution $p \in Q$ which is unique up to $\mathrm{Ker}(B^*)$.
On the other hand, (7.22) holds true, if $\mathrm{Im}(B^*)$ is closed in V^* as follows from the following functional analytic result.

Lemma 7.6 Properties of closed operators
Let V and Q be Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$ and associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$ and assume that $A : V \to V^*$ and $B : V \to Q^*$ are bounded linear operators. Then, the following statements are equivalent:

(i) $\mathrm{Im}(B)$ is closed in Q^*,

(ii) $\mathrm{Im}(B^*)$ is closed in V^*,

(iii) $(\mathrm{Ker}(B))^0 \ = \ \mathrm{Im}(B^*)$,

(iv) $(\mathrm{Ker}(B^*))^0 \ = \ \mathrm{Im}(B)$,

(v) There exists $\beta > 0$ such that for any $g \in \mathrm{Im}(B)$, there exists $v_g \in V$ with $Bv_g = g$ and

$$\|v_g\|_V \ \leq \ \frac{1}{\beta} \, \|g\|_{Q^*} \ , \tag{7.23}$$

(vi) There exists $\beta > 0$ such that for any $f \in \mathrm{Im}(B^*)$, there exists $q_f \in Q$ with $B^* q_f = f$ and

$$\|q_f\|_Q \ \leq \ \frac{1}{\beta} \, \|f\|_{V^*} \ , \tag{7.24}$$

Proof. We refer to [5]. □

Lemma 7.7 inf-sup conditions
Under the assumptions of the previous lemma, there holds:
Statement **(v)** in Lemma 7.6 is equivalent to the **inf-sup condition**

$$\inf_{v \in V \setminus Ker(B)} \sup_{q \in Q} \frac{b(v,q)}{\|v\|_{V \setminus Ker(B)} \|q\|_Q} \ \geq \ \beta \ . \tag{7.25}$$

Statement **(vi)** in Lemma 7.6 is equivalent to the **inf-sup condition**

$$\inf_{q \in Q \setminus Ker(B^*)} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_{Q \setminus Ker(B^*)}} \ \geq \ \beta \ . \tag{7.26}$$

The conditions (7.25),(7.26) are also known as **Brezzi conditions**.
Proof. The proof is left as an exercise. □

Putting the previous results together, we arrive at the following existence and uniqueness result.

Theorem 7.3 Existence and uniqueness result
Let V and Q be Hilbert spaces and let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ be **bounded bilinear forms** with associated operators $A : V \to V^*$ and $B : V \to Q^*$ such that there holds:

(i) The bilinear form $a(\cdot,\cdot)$ is $\mathrm{Ker}(B)$**-elliptic**, i.e., there exists a constant $\alpha > 0$ such that

$$a(v_0, v_0) \ \geq \ \alpha \ \|v_0\|_V^2 \quad , \quad v_0 \in \mathrm{Ker}(B) \ . \tag{7.27}$$

(ii) The bilinear form $b(\cdot,\cdot)$ satisfies the **Brezzi** condition

$$\inf_{q \in Q \setminus Ker(B^*)} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_{Q \setminus Ker(B^*)}} \ \geq \ \beta \ . \tag{7.28}$$

Then, for any $f \in V^*$ and $g \in \mathrm{Im}(B)$, the **saddle point problem** (7.18) admits a solution $(u,p) \in V \times Q$, where $u \in V$ is uniquely determined and $p \in Q$ is unique up to an element of $\mathrm{Ker}(B^*)$.

7.4 Approximation of saddle point problems

Let V and Q be Hilbert spaces and assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ are continuous bilinear forms with associated bounded linear operators $A : V \to V^*$ and $B : V \to Q^*$.
In this section, we consider the approximation of (7.18) by means of finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$:
Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(u_h, v_h) \ + \ b(v_h, p_h) \ &= \ < f, v_h >_{V^*,V} \quad , \quad v_h \in V_h \ , \\ b(u_h, q_h) \ &= \ < g, q_h >_{Q^*,Q} \quad , \quad q_h \in Q_h \ . \end{aligned} \tag{7.29}$$

We denote by $A_h : V_h \to V_h^*$ and $B_h : V_h \to Q_h^*$ the operators associated with $a(\cdot,\cdot)|_{V_h \times V_h}$ and $b(\cdot,\cdot)|_{V_h \times Q_h}$.
In contrast to variational equations considered before, the existence and uniqueness of a solution $(u_h, p_h) \in V_h \times Q_h$ of (7.29) does not follow from the corresponding result for the infinite dimensional problem (7.18). The reason is that the operator B_h in general does not correspond to the restriction of the operator B to V_h, i.e., in general

$$BV_h \ \not\subset Q_h^* \ .$$

In order to establish the **relationship** between the operator $B : V \to Q^*$ and $B_h : V_h \to Q_h^*$, we denote by $P_{Q_h} : Q \to Q_h$ the **orthogonal projection** onto Q_h. Then, we may identify Q_h^* with a subspace of Q^* according to

$$< g_h^*, q >_{Q^*,Q} = < g_h^*, P_{Q_h} q >_{Q^*,Q} \quad , \quad g_h^* \in Q_h^* \ ,$$

which means we extend g_h^* by zero on $Q_h^\perp$, the orthogonal complement of Q_h in Q.
Likewise, for $g \in Q*$ we define its **projection** onto Q_h^* by

$$< P_{Q_h^*} g, q >_{Q^*,Q} = < g, P_{Q_h} q >_{Q^*,Q} = < \underbrace{P_{Q_h}^*}_{= P_{Q_h^*}} g, q >_{Q^*,Q} \quad , \quad g \in Q* .$$

Hence, we obtain

$$< B_h v_h, q >_{Q^*,Q} = < B_h v_h, P_{Q_h} q >_{Q^*,Q} = b(v_h, P_{Q_h} q) =$$
$$= < B v_h, P_{Q_h} q >_{Q^*,Q} = < P_{Q_h^*} B v_h, q >_{Q^*,Q} \quad , \quad v_h \in V_h .$$

from which we deduce

$$B_h = P_{Q_h^*} B|_{V_h} . \tag{7.30}$$

Consequently, the existence and uniqueness of a solution $(u_h, p_h) \in V_h \times Q_h$ of (7.29) require a proper balancing of the subspaces V_h and Q_h.

Theorem 7.4 Existence and uniqueness result

Let V and Q be Hilbert spaces and let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ be **bounded bilinear forms** with associated operators $A : V \to V^*$ and $B : V \to Q^*$.
Moreover, let $V_h \subset V$ and $Q_h \subset Q$ be finite dimensional subspaces and $A_h : V_h \to V_h^*$ and $B_h : V_h \to Q_h^*$ the operators associated with $a(\cdot,\cdot)|_{V_h \times V_h}$ and $b(\cdot,\cdot)|_{V_h \times Q_h}$.
Suppose that the following conditions are satisfied:

(i) The bilinear form $a(\cdot,\cdot)|_{V_h \times V_h}$ is Ker(B_h)**-elliptic**, i.e., there exists a constant $\alpha_h > 0$ such that

$$a(v_h, v_h) \ge \alpha_h \, \|v_h\|_V^2 \quad , \quad v_h \in \text{Ker}(B_h) . \tag{7.31}$$

(ii) The bilinear form $b(\cdot,\cdot)|_{V_h \times Q_h}$ satisfies the **Babuska-Brezzi-Ladyzhenskaya** condition

$$\inf_{q_h \in Q_h \setminus Ker(B_h^*)} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h/Ker(B_h^*)}} \ge \beta_h > 0 . \tag{7.32}$$

Then, for any $f \in V^*$ and $g \in \text{Im}(B)$, the **saddle point problem** (7.29) admits a solution $(u_h, p_h) \in V_h \times Q_h$, where $u_h \in V_h$ is uniquely determined and $p_h \in Q_h$ is unique up to an element of Ker(B_h^*).

Proof. We refer to [1]. □

The following result establishes an **a priori error estimate** for the global discretization error.

Theorem 7.5 A priori error estimate

Suppose that the assumptions of Theorem 7.3 and Theorem 7.4 are satisfied. Let $(u,p) \in V \times Q$ and $(u_h,p_h) \in V_h \times Q_h$ be the solutions of (7.18) and (7.29), respectively. Then there holds

(7.33) $$\|u - u_h\|_V \le$$

$$\le \left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V + \frac{\|b\|}{\alpha_h} \inf_{q_h \in Q_h} \|p - q_h\|_Q \ ,$$

(7.34)

$$\|p - p_h\|_{Q/Ker(B_h^*)} \le \left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q + \frac{\|a\|}{\beta_h} \|u - u_h\|_V \ .$$

Proof of (7.33). In view of the **Babuska-Brezzi-Ladyzhenskaya condition** (7.32) and Lemma 7.6 (v), the set

$$Z_h(g) := \{ v_h \in V_h \mid b(v_h,q_h) = < g, q_h >_{Q^*,Q} \ , \ q_h \in Q_h \}$$

is non empty. For $w_h \in Z_h(g)$ we have $w_h - u_h \in \mathrm{Ker}\ B_h$, and hence,

(7.35) $$\alpha_h \|w_h - u_h\|_V \le \sup_{v_h \in Ker\ B_h} \frac{a(w_h - u_h, v_h)}{\|v_h\|_V} = \sup_{v_h \in Ker\ B_h} \frac{a(w_h - u, v_h) + a(u - u_h, v_h)}{\|v_h\|_V} = \sup_{v_h \in Ker\ B_h} \frac{a(w_h - u, v_h) - b(v_h, p - p_h)}{\|v_h\|_V} \ .$$

Since $v_h \in \mathrm{Ker}\ B_h$, for any $q_h \in Q_h$

(7.36) $$|b(v_h, p - p_h)| = |b(v_h, p - q_h)| \le \|b\| \ \|v_h\|_V \ \|p - q_h\|_Q \ .$$

Using (7.36) in (7.35) yields

$$\alpha_h \|w_h - u_h\|_V \le \|a\| \ \|u - w_h\|_V + \|b\| \ \|p - q_h\|_Q \ ,$$

whence

(7.37) $$\|u - u_h\|_V \le$$

$$\le \left(1 + \frac{\|a\|}{\alpha_h}\right) \inf_{w_h \in Z_h(g)} \|u - w_h\|_V + \frac{\|b\|}{\alpha_h} \inf_{q_h \in Q_h} \|p - q_h\|_Q \ .$$

It remains to estimate

$$\inf_{w_h \in Z_h(g)} \|u - w_h\|_V \ .$$

For any $v_h \in V_h$, we have $B(u - v_h) \in \mathrm{Im}(B)$, and hence, in view of Lemma 7.7 (v) and Lemma 7.8, there exists $r_h \in V$ such that

(7.38) $$b(r_h, q_h) = b(u - v_h, q_h) \quad , \quad q_h \in Q_h$$

satisfying

$$(7.39) \qquad \|r_h\|_V \;\le\; \frac{1}{\beta_h} \sup_{q_h \in Q_h} \frac{b(u - v_h, q_h)}{\|q_h\|_Q} \;\le\; \frac{1}{\beta_h} \|b\| \; \|u - v_h\|_V \; .$$

In view of $Bu = g$, (7.38) tells us that $w_h := r_h + v_h \in Z_h(g)$. Consequently, using (7.39) we obtain

$$(7.40) \quad \|u - w_h\|_V \;\le\; \|u - v_h\|_V + \|r_h\|_V \;\le\; \Bigl(1 + \frac{\|b\|}{\beta_h}\Bigr) \; \|u - v_h\|_V \; .$$

From (7.40) we readily deduce

$$(7.41) \qquad \inf_{w_h \in Z_h(g)} \|u - w_h\|_V \;\le\; \Bigl(1 + \frac{\|b\|}{\beta_h}\Bigr) \inf_{v_h \in V_h} \|u - v_h\|_V \; .$$

Finally, using (7.41) in (7.37) gives (7.33).

Proof of (7.34). If we subtract the first equation in (7.29) from the first equation in (7.18) (choosing $v = v_h \in V_h$), we get

$$a(u - u_h, v_h) \; + \; b(v_h, p - p_h) \;=\; 0 \quad , \quad v_h \in V_h \; .$$

Then, for any $q_h \in Q_h$

$$b(v_h, q_h - p_h) \;=\; - \; a(u - u_h, v_h) \; - \; b(v_h, p - q_h) \; .$$

The **Babuska-Brezzi-Ladyzhenskaya condition** (7.32) implies

$$\begin{aligned} \|q_h - p_h\|_{Q/Ker(B_h^*)} \;&\le\; \frac{1}{\beta_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h - p_h)}{\|v_h\|_V} \;= \\ &=\; \frac{1}{\beta_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h - p) + a(u_h - u, v_h)}{\|v_h\|_V} \; , \end{aligned}$$

whence

$$(7.42) \quad \|q_h - p_h\|_{Q/Ker(B_h^*)} \;\le\; \frac{1}{\beta_h} \Bigl(\|b\| \; \|p - q_h\|_Q \; + \; \|a\| \; \|u - u_h\|_V\Bigr) \; .$$

Then, (7.34) follows by the triangle inequality. □

Theorem 7.5 is a Céa like result which tells us that the discretization errors $u - u_h$ and $p - p_h$ are bounded by the best approximations of the solutions $u \in V$ and $p \in Q$ of (7.18) by functions in V_h and Q_h, respectively. However, in terms of the constants α_h and β_h, the bound still depends on h.
If we require the bilinear form $a(\cdot,\cdot)|_{V_h \times V_h}$ to be **uniformly** $\mathrm{Ker}(B_h)$-**elliptic** and the bilinear form $b(\cdot,\cdot)|_{V_h \times Q_h}$ to satisfy the **Babuska-Brezzi-Ladyzhenskaya** condition uniformly in h, the dependence on h can be removed:

Theorem 7.6 A priori error estimate

In addition to the assumptions of Theorem 7.5 suppose that there exist constants $\alpha > 0$ and $\beta > 0$ such that:

(i) The bilinear form $a(\cdot,\cdot)|_{V_h \times V_h}$ is **uniformly** Ker(B_h)**-elliptic**, i.e.,

$$a(v_h, v_h) \ \geq \ \alpha \ \|v_h\|_V^2 \quad , \quad v_h \in \mathrm{Ker}(B_h) \ . \tag{7.43}$$

(ii) The bilinear form $b(\cdot,\cdot)|_{V_h \times Q_h}$ satisfies the **Babuska-Brezzi-Ladyzhenskaya** condition

$$\inf_{q_h \in Q_h \setminus Ker(B_h^*)} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h / Ker(B_h^*)}} \ \geq \ \beta \ > \ 0 \ . \tag{7.44}$$

Then, we have

$$\|u - u_h\|_V \ \leq \tag{7.45}$$

$$\leq \ \Big(1 + \frac{\|a\|}{\alpha}\Big) \ \Big(1 + \frac{\|b\|}{\beta}\Big) \ \inf_{v_h \in V_h} \|u - v_h\|_V \ + \ \frac{\|b\|}{\alpha} \ \inf_{q_h \in Q_h} \|p - q_h\|_Q \ ,$$

(7.46)

$$\|p - p_h\|_{Q / Ker(B_h^*)} \ \leq \ \Big(1 + \frac{\|b\|}{\beta}\Big) \ \inf_{q_h \in Q_h} \|p - q_h\|_Q \ + \ \frac{\|a\|}{\beta} \ \|u - u_h\|_V \ .$$

Proof. The proof follows directly from Theorem 7.5. □

7.5 Finite dimensional subspaces of $\mathbf{H}(\mathrm{div}, \Omega)$

We consider conforming finite element approximations of the space $\mathbf{H}(\mathrm{div}, \Omega)$ based on both simplicial and rectangular triangulations $\mathcal{T}_h$ of a simply-connected Lipschitz domain $\Omega \subset \mathbb{R}^d$ in case $d = 2$ and $d = 3$.

For $D \subseteq \overline{\Omega}$, we denote by $P_k(D) \ , \ k \in \mathbb{N}_0$ and $\tilde{P}_k(D) \ , \ k \in \mathbb{N}_0$ the set of polynomials of degree $\leq k$ and the set of homogeneous polynomials of degree k on D. Moreover, $Q_{\ell,m,n}(D) \ , \ \ell, m, n \in \mathbb{N}_0$ refers to the set of polynomials in $(x_1, x_2, x_3)^T \in D$ the maximum degree of which are ℓ in x_1, m in x_2, and n in x_3. If $\ell = m = n = k$, we simply write $Q_k(D)$ instead of $Q_{k,k,k}(D)$. Obvious modifications apply in case $D \subset \mathbb{R}^2$.

7.5.1 Conforming elements for $\mathbf{H}(\mathrm{div}, \Omega)$

Let $\mathcal{T}_h$ be a triangulation of Ω and

$$P_K \ := \ \{ \ \mathbf{q} = (q_1, ..., q_d)^T \ | \ q_i : K \to \mathbb{R} \ , \ 1 \leq i \leq d \ \} \ , \tag{7.47}$$

$$\mathbf{V}_h(\Omega) \ := \ \{ \ \mathbf{q}_h : \bar{\Omega} \to \mathbb{R} \ | \ \mathbf{q}_h|_K \in P_K \ , \ K \in \mathcal{T}_h \ \} \ . \tag{7.48}$$

The following result gives sufficient conditions for $\mathbf{V}_h(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega)$.

Theorem 7.7 Sufficient conditions for conformity
Let $\mathcal{T}_h$ be a triangulation of Ω and let P_K , $K \in \mathcal{T}_h$, and $\mathbf{V}_h(\Omega)$ be given by (7.47) and (7.48), respectively. Assume that

(7.49) $$P_K \subset \mathbf{H}(\mathrm{div}; K) \ , \ K \in \mathcal{T}_h \ ,$$

(7.50) $$[\mathbf{n}\cdot\mathbf{q}|_e] = 0 \quad \text{for all } e \ = \ K_i \cap K_j \in \mathcal{E}_h(\Omega) \ , \ \mathbf{q} \in V_h(\Omega), d = 2 \ ,$$

(7.51) $$[\mathbf{n}\cdot\mathbf{q}|_f] = 0 \quad \text{for all } f \ = \ K_i \cap K_j \in \mathcal{F}_h(\Omega) \ , \ \mathbf{q} \in V_h(\Omega), d = 3 \ ,$$

where $[\mathbf{n}\cdot\mathbf{q}|_e]$ and $[\mathbf{n}\cdot\mathbf{q}|_f]$ denote the jump of $\mathbf{n}\cdot\mathbf{q}$ across e and f, i.e.,

(7.52)
$$[\mathbf{n}\cdot\mathbf{q}|_g] := \mathbf{n}\cdot\mathbf{q}|_{g\cap K_i} - \mathbf{n}\cdot\mathbf{q}|_{g\cap K_j} \ , \ g := e \ (d = 2) \text{ and } g := f \ (d = 3) \ .$$

Then $\mathbf{V}_h(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega)$.

Proof. We prove the result in case $d = 3$. The case $d = 2$ can be shown in literally the same way.
Given $\mathbf{q}_h \in \mathbf{V}_h(\Omega)$, we have to show that $\mathrm{div}\mathbf{q}_h$ is well defined and $\mathrm{div}\mathbf{q}_h \in L^2(\Omega)$. In other words, we have to find $z_h \in L^2(\Omega)$ such that

$$\int_\Omega \mathbf{q}_h \cdot \mathbf{grad}\, \varphi \, d\mathbf{x} \ = \ - \int_\Omega z_h \, \varphi \, d\mathbf{x} \quad , \quad \varphi \in \mathcal{D}(\Omega) \ .$$

In view of (7.49), Green's formula can be applied elementwise:

$$\int_\Omega \mathbf{q}_h \cdot \mathbf{grad}\, \varphi \, d\mathbf{x} \ = \ \sum_{K\in\mathcal{T}_h} \int_K \mathbf{q}_h \cdot \mathbf{grad}\, \varphi \, d\mathbf{x} \ =$$
$$= - \sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}\mathbf{q}_h \, \varphi \, d\mathbf{x} \ + \ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbf{n}\cdot\mathbf{q}_h|_{\partial K} \, \varphi \, d\sigma \ =$$
$$= - \sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}\mathbf{q}_h \, \varphi \, d\mathbf{x} \ + \ \sum_{f\in\mathcal{F}_h(\Omega)} \int_f [\mathbf{n}\cdot\mathbf{q}_h|_f] \, \varphi \, d\sigma \ .$$

Taking advantage of (7.51), the assertion follows for z_h with $z_h|_K := \mathrm{div}\mathbf{q}_h, K \in \mathcal{T}_h$.

7.5.2 Raviart-Thomas elements $\mathbf{RT}_k(K)$

Let us first consider the case of simplicial triangulations $\mathcal{T}_h$ of Ω. For $K \in \mathcal{T}_h$ and $k \in \mathbb{N}_0$ we set

$$R_k(\partial K) := \{ \varphi \in L^2(\partial K) \mid \begin{cases} \varphi|_e \in P_k(e) \ , \ e \in \mathcal{E}_h(K) & , \quad d = 2 \\ \varphi|_f \in P_k(f) \ , \ f \in \mathcal{F}_h(K) & , \quad d = 3 \end{cases} \} .$$

Definition 7.4 Raviart-Thomas elements $\mathbf{RT}_k(K)$

Let K be a d-simplex. The **Raviart-Thomas element** $\mathbf{RT}_k(K), k \in \mathbb{N}_0$, is defined by

$$\mathbf{RT}_k(K) \ = \ P_k(K)^d \ + \ \mathbf{x} \ \tilde{P}_k(K) \ . \tag{7.53}$$

For $\mathbf{q} \in \mathbf{RT}_k(K)$, the degrees of freedom Σ_K are given by

$$\int_{\partial K} \mathbf{q} \cdot \mathbf{n} \ p_k \ d\sigma \quad , \quad p_k \in R_k(\partial K) \ , \tag{7.54}$$

$$\int_K \mathbf{q} \cdot \mathbf{p}_{k-1} \ d\mathbf{x} \quad , \quad \mathbf{p}_{k-1} \in P_{k-1}(K)^d \ . \tag{7.55}$$

We have

$$\dim \mathbf{RT}_k(K) \ = \ \begin{cases} (k+1)(k+3) & , \quad d = 2 \ , \\ \frac{1}{2} \ (k+1)(k+2)(k+4) & , \quad d = 3 \end{cases} \ . \tag{7.56}$$

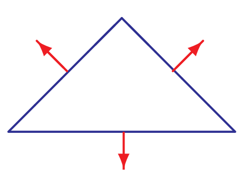

Fig. 7.1: $\mathbf{RT}_0(K)$, $d = 2$

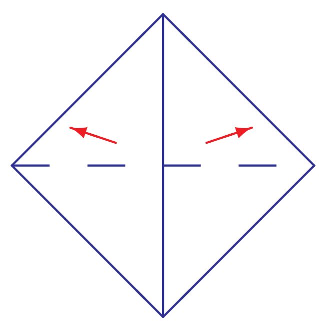

Fig. 7.2: $\mathbf{RT}_0(K)$, $d = 3$

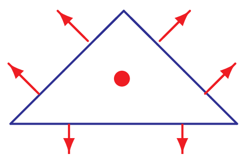

Fig. 7.3: $\mathbf{RT}_1(K)$, $\underset{\cdot}{=}2$

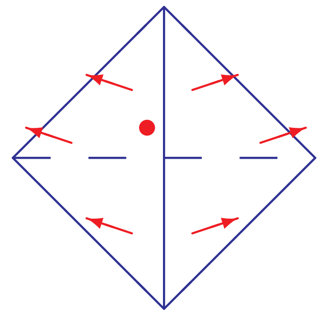

Fig. 7.4: $\mathbf{RT}_1(K)$, $d = 3$

Lemma 7.8 Range of the discrete divergence operator

For $\mathbf{q} \in \mathbf{RT}_k(K)$ we have

(7.57) $$\text{div } \mathbf{q} \in P_k(K) \ ,$$

(7.58) $$\mathbf{n} \cdot \mathbf{q}|_{\partial K} \in R_k(\partial K) \ .$$

Proof. According to (7.54), $\mathbf{q}$ can be written as

(7.59) $$\mathbf{q} \ = \ \mathbf{p}_k \ + \ \mathbf{x} p_k \quad , \quad \mathbf{p}_k \in P_k(K)^d \ , \ p_k \in \tilde{P}_k(K) \ .$$

Then

(7.60) $$\text{div } \mathbf{q} \ = \ \text{div } \mathbf{p}_k \ + \ \mathbf{x} \cdot \mathbf{grad} \ p_k \ + \ 3 \ p_k \ = \ \text{div } \mathbf{p}_k \ + \ (k+3) \ p_k$$

which gives (7.57).
Further, for $\mathbf{n} = (n_1, ..., n_d)^T$ we get

$$\mathbf{n} \cdot \mathbf{q} \ = \ \mathbf{n} \cdot \mathbf{p}_k \ + \ p_k \ \sum_{i=1}^d n_i \ x_i \ .$$

Since $\sum_{i=1}^d n_i \ x_i \ = \ const.$ on $e \subset \partial K$ resp. $f \subset \partial K$, this shows (7.58). □

Lemma 7.9 Auxiliary result for unisolvence

Let $\mathbf{q} \in P_k(K)^d$ and assume

(7.61) $$\int_{\partial K} \mathbf{q} \cdot \mathbf{n} \ p_k \ d\sigma \ = \ 0 \quad , \quad p_k \in R_k(\partial K) \ ,$$

(7.62) $$\int_K \mathbf{q} \cdot \mathbf{p}_{k-1} \ d\mathbf{x} \ = \ 0 \quad , \quad \mathbf{p}_{k-1} \in P_{k-1}(K)^d$$

Then

$$\text{div } \mathbf{q} \ = \ 0 \ .$$

Proof. We apply Green's formula

$$\int_K \operatorname{div} \mathbf{q}\, p_{k-1}\, d\mathbf{x} \;=\; - \underbrace{\int_K \mathbf{q} \cdot \mathbf{grad}\, p_{k-1}\, d\mathbf{x}}_{=0} \;+\; \underbrace{\int_{\partial K} \mathbf{n} \cdot \mathbf{q}\, p_{k-1}\, d\sigma}_{=0}\;,$$

$$p_{k-1} \in P_{k-1}(K)\;.$$

Since $\operatorname{div} \mathbf{q} \in P_{k-1}(K)$, we may choose $p_{k-1} = \operatorname{div} \mathbf{q}$ which gives the assertion. □

Theorem 7.8 Unisolvence of the $\mathbf{RT}_k(K)$ element

The element $(K, \mathbf{RT}_k(K), \Sigma_K)$ is **unisolvent**.

Proof. We have to show that the relations (7.61) and (7.62) imply $\mathbf{q} = 0$.
Due to (7.57) we have $\mathbf{n}\cdot\mathbf{q} \in R_k(\partial K)$ and hence, (7.61) implies $\mathbf{n}\cdot\mathbf{q} = 0$ on each edge resp. face.
In the same way as in Lemma 7.9 we deduce $\operatorname{div} \mathbf{q} = 0$. Observing Lemma 7.9 and (7.60), we get $p_k = 0$.
Taking advantage of the affine equivalence, we consider the reference tetrahedron (cf. Figure 7.5).

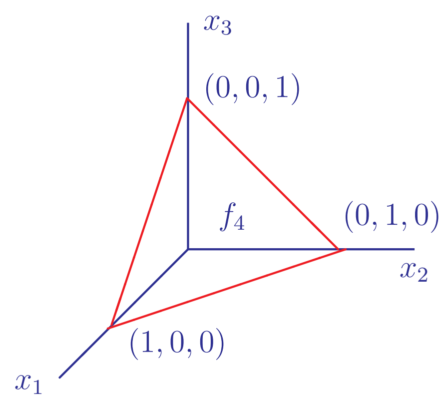

Fig. 7.5: The reference tetrahedron

In lights of the results obtained so far, we may assume

$$\hat{q}_i \;=\; \hat{x}_i\, \hat{\psi}_i \quad , \quad \hat{\psi}_i \in P_{k-1}(\hat{K})\;,\; 1 \le i \le 3\;.$$

If we choose $\hat{\mathbf{p}}_{k-1} := (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)^T$ in (7.60), we obtain

$$\sum_{i=1}^{3} \int_{\hat{K}} \hat{x}_i\, \hat{\psi}_i^2\, d\mathbf{x} \;=\; 0$$

whence $\hat{\psi}_i = 0 \ , \ 1 \le i \le 3$, and thus $\hat{\mathbf{q}} = 0$. □

Definition 7.5 The Raviart-Thomas finite element space $\mathbf{RT}_k(\Omega, \mathcal{T}_h)$

The **Raviart-Thomas finite element space** $\mathbf{RT}_k(\Omega, \mathcal{T}_h)$ is given by

(7.63) $$\mathbf{RT}_k(\Omega, \mathcal{T}_h) \ := \ \{ \ \mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega) \mid \mathbf{q}|_K \in \mathbf{RT}_k(K) \ , \ K \in \mathcal{T}_h \ \} \ .$$

It is a finite dimensional subspace of $\mathbf{H}(\mathrm{div};\Omega)$.

7.5.3 Raviart-Thomas elements $\mathbf{RT}_{[k]}(K)$

Let $\mathcal{T}_h$ be a rectangular triangulation of Ω. In much the same way as before we define

(7.64)
$$Q_k(\partial K) \ := \ \{\mathbf{q} \in L^2(K) \mid \left\{ \begin{array}{lcl} \mathbf{q}|_e \in P_k(e)^2 \ , \ e \in \mathcal{E}_h(K) & , & d = 2 \\ \mathbf{q}|_f \in Q_k(f) \ , \ f \in \mathcal{F}_h(K) & , & d = 3 \end{array} \right\} \ .$$

Moreover, we set

(7.65)
$$\Psi_k(K) \ := \ \left\{ \begin{array}{rcl} Q_{k-1,k}(K) \times Q_{k,k-1}(K) & , & d = 2 \ , \\ Q_{k-1,k,k}(K) \times Q_{k,k-1,k}(K) \times Q_{k,k,k-1}(K) & , & d = 3 \end{array} \right. \ .$$

Definition 7.6 Raviart-Thomas elements $\mathbf{RT}_{[k]}(K)$

Let K be a d-rectangle. The **Raviart-Thomas element** $\mathbf{RT}_{[k]}(K) \ , \ k \in \mathbb{N}_0$, is defined by

(7.66) $$\mathbf{RT}_{[k]}(K) \ = \ Q_k(K)^d \ + \ \mathbf{x} \ Q_k(K) \ .$$

For $\mathbf{q} \in RT_{[k]}(K)$, the degrees of freedom Σ_K are given by

(7.67) $$\int_{\partial K} \mathbf{q} \cdot \mathbf{n} \ d\sigma \quad , \quad \mathbf{p}_k \in Q_k(\partial K) \ ,$$

(7.68) $$\int_K \mathbf{q} \cdot \mathbf{p}_k \ d\mathbf{x} \quad , \quad \mathbf{p}_k \in \Psi_k(K) \ .$$

We have

(7.69) $$\dim \ \mathbf{RT}_{[k]}(K) \ = \ \left\{ \begin{array}{rcl} 2 \ (k+1)(k+2) & , & d = 2 \ , \\ 3 \ (k+1)^2(k+2) & , & d = 3 \end{array} \right. \ .$$

Theorem 7.9 Unisolvence of the $\mathbf{RT}_{[k]}(K)$ element

The element $(K, \mathbf{RT}_{[k]}(K), \Sigma_K)$ is **unisolvent**.

Proof. The proof is left as an exercise. □

Definition 7.7 The Raviart-Thomas finite element space $\mathbf{RT}_{[k]}(\Omega, \mathcal{T}_h)$
The **Raviart-Thomas finite element space** $\mathbf{RT}_k(\Omega, \mathcal{T}_h)$ is given by

$$\mathbf{RT}_{[k]}(\Omega, \mathcal{T}_h) := \{\mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega) \mid \mathbf{q}|_K \in \mathbf{RT}_{[k]}(K), K \in \mathcal{T}_h\} \ . \quad (7.70)$$

It is a finite dimensional subspace of $\mathbf{H}(\mathrm{div};\Omega)$.

7.6 A priori error estimates for Raviart-Thomas approximations

The abstract a priori error estimates in Theorem 7.6 allow us to derive a priori error estimates for Raviart-Thomas approximations by **interpolation** in $\mathbf{H}(\mathrm{div}, \Omega)$.
We will also strongly take advantage of the **affine equivalence** of the elements and therefore, we have to study how a vector field and its divergence transforms under an affine mapping.

7.6.1 Change of variables

Let $\hat{K}$ be the reference d-simplex and $K = F(\hat{K})$ under the affine mapping $F : \mathbb{R}^d \to \mathbb{R}^d, F(\hat{x}) = B\hat{x} + b$, with $B \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$.
If $\hat{\mathbf{q}} \in L^2(\hat{K})^d$, the transformation

$$q_i \ := \ \hat{q}_i \circ F^{-1} \quad , \quad 1 \le i \le d \ ,$$

does not preserve normal components and does not map $\mathbf{H}(div, \hat{K})$ into $\mathbf{H}(div, K)$. Instead, we use the so-called **Piola transformation**.

Definition 7.8 Piola transformation
Let $F : \mathbb{R}^d \to \mathbb{R}^d, F(\hat{x}) = B\hat{x} + b$, with $B \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$ be an affine mapping such that $K = F(\hat{K}$.
The mapping

$$\begin{aligned} (7.71) \quad \mathcal{P} : L^2(\hat{K})^d \quad &\to \quad L^2(K)^d \\ \hat{q} \quad &\longmapsto \quad \mathbf{q}(x) = \mathcal{P}(\hat{\mathbf{q}})(x) := \frac{1}{det(B)} \ B\hat{\mathbf{q}}(\hat{x}) \end{aligned}$$

is called the **Piola transformation**.

Lemma 7.10 Properties of the Piola transformation, Part I
Let $\mathcal{P}$ be the Piola transformation as given by (7.71). Then, if $\hat{\mathbf{q}} \in C^1(\hat{K})^d$, we have $\mathbf{q} = \mathcal{P}\hat{\mathbf{q}} \in C^1(K)^d$, and there holds

$$\mathbf{Dq} \ = \ \frac{1}{det(B)} \ B \ \hat{\mathbf{D}}\hat{\mathbf{q}} \ B^{-1} \ , \quad (7.72)$$

$$\text{div } \mathbf{q} \; = \; \frac{1}{det(B)} \, \widehat{\text{div}} \; \hat{\mathbf{q}} \; , \tag{7.73}$$

where $\hat{\mathbf{D}}\hat{\mathbf{q}}$ and $\mathbf{Dq}$ denote the Jacobians of $\hat{\mathbf{q}}$ and $\mathbf{q}$, respectively.

Proof. (7.72) can be easily established by elementary calculations, whereas (7.73) follows from the fact that the trace of a matrix is invariant with respect to a change of variables. □

Lemma 7.11 Properties of the Piola transformation, Part II
Let $\hat{v} \in H^1(\hat{K})$, $\hat{q} \in \mathbf{H}(\widehat{div}, \hat{K})$ and $v := \hat{v} \circ F^{-1}$, $q := \mathcal{P}(\hat{q})$. Then, there holds

$$\int_K \mathbf{q} \cdot \mathbf{grad} \; v \; dx \; = \; \int_{\hat{K}} \hat{\mathbf{q}} \cdot \widehat{\mathbf{grad}} \; \hat{v} \; d\hat{x} \; , \tag{7.74}$$

$$\int_K \text{div } \mathbf{q} \; v \; dx \; = \; \int_{\hat{K}} \widehat{\text{div}} \; \hat{q} \; \hat{v} \; d\hat{x} \; , \tag{7.75}$$

$$\int_{\partial K} \mathbf{q} \cdot \mathbf{n} \; v \; d\sigma \; = \; \int_{\partial \hat{K}} \hat{q} \cdot \hat{\mathbf{n}} \; \hat{v} \; d\hat{\sigma} \; . \tag{7.76}$$

Proof. We refer to [4]. □

In particular, (7.76) shows that the Piola transformation preserves normal traces in $H^{-1/2}$. In fact, the Piola transformation is suitable for affine transformations involving vector fields in $\mathbf{H}(div)$:

Lemma 7.12 Properties of the Piola transformation, Part III
The Piola transformation maps $\mathbf{H}(\widehat{\text{div}}, \hat{K})$ isomorphically onto $\mathbf{H}(\text{div}, K)$ and there holds

$$\|\mathbf{q}\|_{0,K} \; \leq \; |\det(B)|^{-1/2} \; \|B\| \; \|\hat{\mathbf{q}}\|_{0,\hat{K}} \; , \tag{7.77}$$

$$\|\hat{\mathbf{q}}\|_{0,\hat{K}} \; \leq \; |\det(B)|^{1/2} \; \|B^{-1}\| \; \|\mathbf{q}\|_{0,K} \; , \tag{7.78}$$

$$\|\text{div } \mathbf{q}\|_{0,K} \; \leq \; |\det(B)|^{-1/2} \; \|\widehat{\text{div}} \; \hat{\mathbf{q}}\|_{0,\hat{K}} \; , \tag{7.79}$$

$$\|\widehat{\text{div}} \; \hat{\mathbf{q}}\|_{0,\hat{K}} \; \leq \; |\det(B)|^{1/2} \; \|\text{div } \mathbf{q}\|_{0,K} \; . \tag{7.80}$$

Proof. We refer to [4]. □

If we define

$$\mathbf{H}^m(\text{div}, \Omega) \; := \; \{\mathbf{q} \in H^m(\Omega)^d \mid \text{div } \mathbf{q} \in H^m(\Omega)\} \quad , \quad m \in \mathbb{N} \; , \tag{7.81}$$

we have the following transformation rules:

Lemma 7.12 Properties of the Piola transformation, Part IV
The Piola transformation maps $\mathbf{H}^m(\widehat{\text{div}}, \hat{K}), m \in \mathbb{N}$, isomorphically onto $\mathbf{H}^m(\text{div}, K)$ and there holds

$$\|\mathbf{q}\|_{m,K} \le |\det(B)|^{-1/2} \|B^{-1}\|^m \|B\| \|\hat{\mathbf{q}}\|_{m,\hat{K}} , \tag{7.82}$$

$$\|\hat{\mathbf{q}}\|_{m,\hat{K}} \le |\det(B)|^{1/2} \|B\|^m \|B^{-1}\| \|\mathbf{q}\|_{m,K} , \tag{7.83}$$

$$\|\text{div } \mathbf{q}\|_{m,K} \le |\det(B)|^{-1/2} \|B^{-1}\|^m \|\widehat{\text{div}} \hat{\mathbf{q}}\|_{m,\hat{K}} , \tag{7.84}$$

$$\|\widehat{\text{div}} \hat{\mathbf{q}}\|_{m,\hat{K}} \le |\det(B)|^{1/2} \|B\|^m \|\text{div } \mathbf{q}\|_{m,K} . \tag{7.85}$$

Proof. We refer to [4]. □

7.6.2 Estimation of the local interpolation error

The degrees of freedom of a vector field $\mathbf{q} \in \mathbf{RT}_k(K), k \in \mathbb{N}_0$, are defined by means of functions $p_k \in R_k(\partial K)$ which belong to $H^s(\partial K), s < 1/2$. Since $\mathbf{n} \cdot \mathbf{q} \in H^{-1/2}(\partial k), \mathbf{q} \in \mathbf{H}(\text{div}, K),$, it is clear that for the proper definition of a **local interpolation operator** we need some more **regularity**.

Definition 7.9 Local interpolation operator
Let K be a non degenerate d-simplex and $k \in \mathbb{N}_0$. The **local interpolation operator**

$$\rho_K : \mathbf{H}(\text{div}, K) \cap L^s(K)^d \to \mathbf{RT}_k(K) \quad , \quad s > 2 , \tag{7.86}$$

is defined as follows:

$$\int_{\partial K} \Big(\mathbf{q} - \rho_K \mathbf{q}\Big) \cdot \mathbf{n} \, p_k \, d\sigma = 0 \quad , \quad p_k \in R_k(\partial K) , \tag{7.87}$$

$$\int_{K} \Big(\mathbf{q} - \rho_K \mathbf{q}\Big) \cdot \mathbf{p}_{k-1} \, dx = 0 \quad , \quad \mathbf{p}_{k-1} \in P_{k-1}(K)^d .$$

Theorem 7.10 Estimation of the local interpolation error
Let K be a non degenerate d-simplex and let $\rho_{\hat{K}} : \mathbf{H}(div, K) \cap L^s(K)^d \to \mathbf{RT}_k(K), k \in \mathbb{N}_0$, be the **local interpolation operator**. Then, there exists a positive constant $C(k, K)$ such that for $\mathbf{q} \in H^m(K)^d, 1 \le m \le k+1$, there holds

$$\|\mathbf{q} - \rho_K \mathbf{q}\|_{\ell,K} \le C(k,K) \, h_K^{m-\ell} \, |\mathbf{q}|_{m,K} \quad , \quad 0 \le \ell \le 1 . \tag{7.88}$$

Proof. The proof can be done using Lemma 7.12 and standard scaling arguments. □

In order to study the error in $\mathbf{H}(\mathrm{div}, K)$, we need the following important **commuting diagram property**:

Lemma 7.13 Commuting diagram property
Let K be a non degenerate d-simplex, ρ_K the interpolation operator given by (7.87), and π_K the L^2-projection onto $\mathrm{div}\ \mathbf{RT}_k(K)$.
Then, for all $\mathbf{q} \in \mathbf{H}(\mathrm{div}, K) \cap L^s(K)^d, s > 2,$ there holds

(7.89) $$\mathrm{div}(\mathbf{q}) \;=\; \pi_K(\mathrm{div}\ \mathbf{q}) \ .$$

Note that (7.89) is equivalent to the **commuting diagram property**

(7.90)
$$\begin{array}{ccc} & \mathrm{div} & \\ \mathbf{H}(\mathrm{div}, K) \cap L^s(K)^d & \longrightarrow & L^2(K) \\ \rho_k \quad \downarrow & & \downarrow \quad \pi_K \\ \mathbf{RT}_k(K) & \longrightarrow & \mathrm{div}\ \mathbf{RT}_k(K) \\ & \mathrm{div} & \end{array}$$

Proof. Since $\pi_k\ \mathrm{div}(\mathbf{q}) \in \mathrm{div}\ \mathbf{RT}_k(K)$, it suffices to prove

$$\int_K v\ \mathrm{div}(\rho_K \mathbf{q})\ dx \;=\; \int_K v\ \mathrm{div}(\mathbf{q})\ dx \quad , \quad v \in \mathrm{div}\ \mathbf{RT}_k(K) \ .$$

Using (7.88), we find

$$\int_K v\ \Big(\mathrm{div}(\rho_K \mathbf{q}) - \mathrm{div}(\mathbf{q})\Big)\ dx \;=$$
$$= \int_K \mathbf{grad}\ v \cdot \Big(\mathbf{q} - \rho_K \mathbf{q}\Big)\ dx \;-\; \int_{\partial K} \mathbf{n} \cdot v\ \Big(\mathbf{q} - \rho_K \mathbf{q}\Big)\ d\sigma \;=\; 0 \ ,$$

which gives the assertion. □

Theorem 7.11 Estimation of the local interpolation error
Let K be a non degenerate d-simplex and let $\rho_{\hat{K}} : \mathbf{H}(div, \Omega) \cap L^s(\Omega)^d \rightarrow \mathbf{RT}_k(K), k \in \mathbb{N}_0,$ be the **local interpolation operator**. Then, there exists a positive constant $C(k, K)$ such that for $\mathbf{q} \in \mathbf{H}^m(\mathrm{div}, K), 1 \leq m \leq k+1,$ there holds

(7.91) $$\|\mathrm{div}(\mathbf{q} - \rho_K \mathbf{q})\|_{0,K} \;\leq\; C(k, K)\ h_K^m\ |\mathrm{div}(\mathbf{q})|_{m,K} \ .$$

Proof. The proof can be done using Lemmas 7.11,7.12 and standard scaling arguments. □

7.6.3 Estimation of the global interpolation error

We are now in a position to estimate the **global interpolation error**. For that purpose, we have to specify the **global interpolation operator**.

Definition 7.10 Global interpolation operator

Let $\mathcal{T}_h(\Omega)$ be a shape regular simplicial triangulation of $\Omega \subset \mathbb{R}^d$. The **global interpolation operator**

$$\Pi_h \ : \ \mathbf{H}(\mathrm{div},\Omega) \cap L^s(\Omega)^d \to \mathbf{RT}_k(\Omega,\mathcal{T}_h(\Omega)) \quad , \quad s > 2 \ , \tag{7.92}$$

is defined as follows:

$$(\Pi_h(\mathbf{q}))|_K \ := \ \rho_K(\mathbf{q}|_K) \quad , \quad K \in \mathcal{T}_h(\Omega) \ . \tag{7.93}$$

Lemma 7.14 Commuting diagram property

Let $\mathcal{T}_h(\Omega)$ be a shape regular simplicial triangulation of $\Omega \subset \mathbb{R}^d$ and denote by Π_h the global interpolation operator and by P_h the L^2-projection onto

$$\begin{aligned} W^0(\mathrm{div}\ \mathbf{RT}_k(\Omega,\mathcal{T}_h),\mathcal{T}_h) \ &:= \\ := \ \{v \in L^2(\Omega) \mid v|_K \in \mathrm{div}\ \mathbf{RT}_k(K) \ , \ K \in \mathcal{T}_h(\Omega)\} \ . \end{aligned} \tag{7.94}$$

Then, the following **commuting diagram property** holds true

$$\begin{array}{ccc} & \mathrm{div} & \\ \mathbf{H}(\mathrm{div},\Omega) \cap L^s(\Omega)^d & \longrightarrow & L^2(\Omega) \\ \Pi_h \quad \downarrow & & \downarrow \quad P_h \\ \mathbf{RT}_k(\Omega,\mathcal{T}_h) & \longrightarrow & W^0(\mathrm{div}\ \mathbf{RT}_k(\Omega,\mathcal{T}_h),\mathcal{T}_h) \\ & \mathrm{div} & \end{array} \tag{7.95}$$

Proof. The proof is left as an exercise. □

Theorem 7.12 Estimation of the global interpolation error

Let $\mathcal{T}_h(\Omega)$ be a shape regular simplicial triangulation of $\Omega \subset \mathbb{R}^d$ and denote by Π_h the global interpolation operator. Then, there exists a positive constant C, independent of the mesh size h, such that for $\mathbf{q} \in \mathbf{H}^{k+1}(\mathrm{div},\Omega)$ and $1 \leq m \leq k+1$

$$\|\mathbf{q} - \Pi_h \mathbf{q}\|_{0,\Omega} \ \leq \ C \ h^m \ |\mathbf{q}|_{m,\Omega} \ , \tag{7.96}$$

$$\|\mathrm{div}(\mathbf{q} - \Pi_h \mathbf{q})\|_{0,\Omega} \ \leq \ C \ h^\ell \ |\mathrm{div}(\mathbf{q})|_{\ell,\Omega} \quad , \quad \ell \leq k+1 \ . \tag{7.97}$$

Proof. The proof follows readily from Theorems 7.10,7.11. □

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