Chapter 1: Financial Markets and Financial Derivatives

1.1 Financial Markets

Financial markets are markets for financial instruments, in which buyers and sellers find each other and create or exchange financial assets.

- Financial instruments
  A financial instrument is a real or virtual document having legal force and embodying or conveying monetary value.

- Financial assets
  A financial asset is an asset whose value does not arise from its physical embodiment but from a contractual relationship.

Typical financial assets are bonds, commodities, currencies, and stocks.

Financial markets may be categorized as either money markets or capital markets.
Money markets deal in short term debt instruments whereas capital markets trade in long term debt and equity instruments.
1.2 Financial Derivatives

A financial derivative is a contract between individuals or institutions whose value at the maturity date (or expiry date) $T$ is uniquely determined by the value of an underlying asset (or assets) at time $T$ or until time $T$.

We distinguish three classes of financial derivatives:

(i) Options

Options are contracts that give the holder the right (but not the obligation) to exercise a certain transaction on the maturity date $T$ or until the maturity date $T$ at a fixed price $K$, the so-called exercise price (or strike).

(ii) Forwards and Futures

A forward is an obligatory contract to buy or sell an asset on the maturity date $T$ at a fixed price $K$. A future is a standardized forward whose value is computed on a daily basis.

(iii) Swaps

A swap is a contract to exercise certain financial transactions at fixed time instants according to a prescribed formula.
1.3 Options

The basic options are the so-called **plain-vanilla options**.
We distinguish between the right to buy or sell assets:

- **Call or call-options**
  A call (or a call-option) is a contract between a **holder** (the buyer) and a **writer** (the seller) which gives the holder the right to buy a financial asset from the writer on or until the maturity date $T$ at a fixed strike price $K$.

- **Put or put-option**
  A put (or a put-option) is a contract between a **holder** (the seller) and a **writer** (the buyer) which gives the holder the right to sell a financial asset to the writer on or until the maturity date $T$ at a fixed strike price $K$. 
1.4 European and American Options, Exotic Options

- European options
A European call-option (put-option) is a contract under the following condition: On the maturity date $T$ the holder has the right to buy from the writer (sell to the writer) a financial asset at a fixed strike price $K$.

- American options
An American call-option (put-option) is a contract under the following condition: The holder has the right to buy from the writer (sell to the writer) a financial asset until the maturity date $T$ at a fixed strike price $K$.

- Exotic options
The European and American options are called standard options. All other (non-standard) options are referred to as exotic options. The main difference between standard and non-standard options is in the payoff.
1.5 European Options: Payoff Function

Since an option gives the holder a right, it has a value which is called the option price.

- **Call-option**
  We denote by $C_t = C(t)$ the value of a call-option at time $t$ and by $S_t = S(T)$ the value of the financial asset at time $t$. We distinguish two cases:

  (i) At the maturity date $T$, the value $S_T$ of the asset is higher than the strike price $K$. The call-option is then exercised, i.e., the holder buys the asset at price $K$ and immediately sells it at price $S_T$. The holder realizes the profit $V(S_T, T) = C_T = S_T - K$.

  (ii) At the maturity date $T$, the value $S_T$ of the asset is less than or equal to the strike price. In this case, the holder does not exercise the call-option, i.e., the option expires worthless with $V(S_T, T) = C_T = 0$.

In summary, at the maturity date $T$ the value of the call is given by the payoff function

$$V(S_T, T) = (S_T - K)^+ := \max\{S_T - K, 0\}.$$
• Put-option

We denote by $P_t = P(t)$ the value of a put-option at time $t$ and by $S_t = S(T)$ the value of the financial asset at time $t$. We distinguish the cases:

(i) At the maturity date $T$, the value $S_T$ of the asset is less than the strike price $K$. The put-option is then exercised, i.e., the holder buys the asset for the market price $S_T$ and sells it to the writer at price $K$. The holder realizes the profit $V(S_T, T) = P_T = K - S_T$.

(ii) At the maturity date $T$, the value $S_T$ of the asset is greater than or equal to the strike price. In this case, the holder does not exercise the put-option, i.e., the option expires worthless with $V(S_T, T) = P_T = 0$.

In summary, at the maturity date $T$ the value of the put is given by the payoff function

$$V(S_T, T) = (K - S_T)^+ := \max\{K - S_T, 0\}.$$
Payoff Function of a European Call and a European Put

European Call

European Put
Example: Call-options
A company A wants to purchase 20,000 stocks of another company B in six months from now. Assume that at present time \( t = 0 \) the value of a stock of company B is \( S_0 = 90 \)$. 
The company A does not want to spend more than 90$ per stock and buys 200 call-options with the specifications \( K = 90 \), \( T = 6 \), \( C_0 = 500 \), 
where each option gives the right to purchase 100 stocks of company B at a price of 90$ per stock.

If the price of the stock on maturity date \( T = 6 \) is \( S_T > 90 \)\$, company A will exercise the option and spend 1,8 Mio $ for the stocks and \( 200 \cdot C_0 = 100,000 \) $ for the options.
Company A has thus insured its purchase against the volatility of the stock market.

On the other hand, company A could have used the options to realize a profit. For instance, if on maturity date \( T = 6 \) the market price is \( S_T = 97 \)\$, the company could buy the 200,000 stocks at a price of 1,8 Mio $ and immediately sell them at a price of 97$ per stock which makes a profit of \( 7 \times 20,000 - 100,000 = 40,000 \) $.
However, if \( S_T < 90 \)\$, the options expire worthless, and A realizes a loss of 100,000$. 
Example: Arbitrage
Consider a financial market with three different financial assets: a bond, a stock, and a call-option with $K = 100$ and maturity date $T$. We recall that a bond $B$ with value $B_t = B(t)$ is a risk-free asset which is paid for at time $t = 0$ and results in $B_T = B_0 + i_R B_0$, where $i_R$ is a fixed interest rate. At time $t = 0$, we assume $B_0 = 100, S_0 = 100$ and $C_0 = 10$. We further assume $i_R = 0.1$ and that at $T$ the market attains one of the two possible states

- 'high': $B_T = 110$, $S_T = 120$
- 'low': $B_T = 110$, $S_T = 80$

A clever investor chooses a portfolio as follows: He buys $\frac{2}{5}$ of the bond and $1$ call-option and sells $\frac{1}{2}$ stock. Hence, at time $t = 0$ the portfolio has the value

$$\pi_0 = \frac{2}{5} \cdot 100 + 1 \cdot 10 - \frac{1}{2} \cdot 100 = 0,$$

i.e., no costs occur for the investor. On maturity date $T$, we have

- 'high': $\pi_T = \frac{2}{5} \cdot 110 + 1 \cdot 20 - \frac{1}{2} \cdot 120 = 4$
- 'low': $\pi_T = \frac{2}{5} \cdot 110 + 1 \cdot 0 - \frac{1}{2} \cdot 80 = 4$

Since for both possible states the portfolio has the value $4$, the investor could sell it at time $t = 0$ and realize an immediate, risk-free profit called arbitrage.
Example: No-Arbitrage (Duplication Strategy)

The reason for the arbitrage in the previous example is due to the fact that the price for the call-option is too low. Therefore, the question comes up: What is an appropriate price for the call to exclude arbitrage?

We have to assume that a portfolio consisting of a bond $B_t$ and a stock $S_t$ has the same value as the call-option, i.e., that there exist numbers $c_1 > 0, c_2 > 0$ such that at $t = T$:

$$c_1 \cdot B_T + c_2 \cdot S_T = C(S_T, T).$$

Hence, the 'fair' price for the call-option is given by

$$p = c_1 \cdot B_0 + c_2 \cdot S_0.$$

Recalling the previous example, at time $t = T$ we have

'high' $c_1 \cdot 110 + c_2 \cdot 120 = 20,$

'low' $c_1 \cdot 110 + c_2 \cdot 80 = 0.$

The solution of this linear system is $c_1 = -\frac{4}{11}, c_2 = \frac{1}{2}.$ Consequently, the fair price is

$$p = -\frac{4}{11} \cdot 100 + \frac{1}{2} \cdot 100 = \frac{300}{22} \approx 13.64.$$
1.6 No-Arbitrage and Put-Call Parity

We consider a financial market under the following assumptions:

- There is no-arbitrage.
- There is no dividend on the basic asset.
- There is a fixed interest rate $r > 0$ for bonds/credits with proportional yield.
- The market is liquid and trade is possible any time.
Reminder: Interest with Proportional Yield

At time $t = 0$ we invest the amount of $K_0$ in a bond with interest rate $r > 0$ and proportional yield, i.e., at time $t = T$ the value of the bond is

$$K = K_0 \exp(rT).$$

In other words, in order to obtain the amount $K$ at time $t = T$ we must invest

$$K_0 = K \exp(-rT).$$

This is called discounting.
Theorem 1.1 Put-Call Parity

Let $K, S_t, P_E(S_t, t)$ and $C_E(S_t, t)$ be the values of a bond (with interest rate $r > 0$ and proportional yield), an asset, a European put, and a European call. Under the previous assumptions, for $0 \leq t \leq T$ there holds

$$\pi_t := S_t + P_E(S_t, t) - C_E(S_t, t) = K \exp(-r(T - t)).$$

Proof. First, assume $\pi_t < K \exp(-r(T - t)).$ Buy the portfolio, take the credit $K \exp(-r(T - t))$ (or sell corresponding bonds) and save the amount $K \exp(-r(T - t)) - \pi_t > 0.$ On maturity date $T,$ the portfolio has the value $\pi_T = K$ which is given to the bank for the credit. This means that at time $t$ a risk-free profit $K \exp(-r(T - t)) - \pi_t > 0$ has been realized contradicting the no-arbitrage principle.

Now, assume $\pi_t > K \exp(-r(T - t)).$ Sell the portfolio (i.e., sell the asset and the put and buy a call), invest $K \exp(-r(T - t))$ in a risk-free bond and save $\pi_t - K \exp(-r(T - t)) > 0.$ On maturity date $T,$ get $K$ from the bank and buy the portfolio at price $\pi_T = K.$ This means a risk-free profit $\pi_t - K \exp(-r(T - t)) > 0$ contradicting the no-arbitrage principle.
**Arbitrage Table for the Proof of Theorem 1.1**

The proof of the put-call parity can be illustrated by the following arbitrage tables

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Cash Flow</th>
<th>Value Portfolio at t</th>
<th>Value Portfolio at T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy $S_t$</td>
<td>$-S_t$</td>
<td>$S_t$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Buy $P_E(S_t,t)$</td>
<td>$-P_E(S_t,t)$</td>
<td>$P_E(S_t,t)$</td>
<td>$K - S_T$</td>
</tr>
<tr>
<td>Sell $C_E(S_t,t)$</td>
<td>$C_E(S_t,t)$</td>
<td>$-C_E(S_t,t)$</td>
<td>$0$</td>
</tr>
<tr>
<td>Credit $K \exp(-r(T-t))$</td>
<td>$K \exp(-r(T-t))$</td>
<td>$-K \exp(-r(T-t))$</td>
<td>$-K$</td>
</tr>
<tr>
<td>Sum</td>
<td>$K \exp(-r(T-t)) - \pi_t &gt; 0$</td>
<td>$-K \exp(-r(T-t)) + \pi_t &lt; 0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
## Arbitrage Table for the Proof of Theorem 1.1

Arbitrage table for the second part of the proof of Theorem 1.1.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Cash Flow</th>
<th>Value Portfolio at $t$</th>
<th>Value Portfolio at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell $S_t$</td>
<td>$S_t$</td>
<td>$-S_t$</td>
<td>$-S_T$</td>
</tr>
<tr>
<td>Sell $P_E(S_t, t)$</td>
<td>$P_E(S_t, t)$</td>
<td>$-P_E(S_t, t)$</td>
<td>$-(K - S_T)$</td>
</tr>
<tr>
<td>Buy $C_E(S_t, t)$</td>
<td>$-C_E(S_t, t)$</td>
<td>$C_E(S_t, t)$</td>
<td>0</td>
</tr>
<tr>
<td>Invest $K \exp(-r(T - t))$</td>
<td>$-K \exp(-r(T - t))$</td>
<td>$K \exp(-r(T - t))$</td>
<td>$K$</td>
</tr>
<tr>
<td>Sum</td>
<td>$\pi_t - K \exp(-r(T - t)) &gt; 0$</td>
<td>$K \exp(-r(T - t)) - \pi_t &lt; 0$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$S_T \leq K$</th>
<th>$S_T &gt; K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell $S_t$</td>
<td>$-S_T$</td>
<td>$-S_T$</td>
</tr>
<tr>
<td>Sell $P_E(S_t, t)$</td>
<td>$-(K - S_T)$</td>
<td>0</td>
</tr>
<tr>
<td>Buy $C_E(S_t, t)$</td>
<td>0</td>
<td>$S_T - K$</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Theorem 1.2  Lower and Upper Bounds for European Options

Let $K, S_t, P_E(S_t, t)$ and $C_E(S_t, t)$ be the values of a bond (with interest rate $r > 0$ and proportional yield), an asset, a European put, and a European call. Under the previous assumptions, for $0 \leq t \leq T$ there holds

\begin{align*}
(\ast) & \quad (S_t - K \exp(-r(T - t)))^+ \leq C_E(S_t, t) \leq S_t, \\
(\ast\ast) & \quad (K \exp(-r(T - t)) - S_t)^+ \leq P_E(S_t, t) \leq K \exp(-r(T - t)).
\end{align*}

Proof of $(\ast)$ . Obviously, $C_E(S_t, t) \geq 0$, since otherwise the purchase of the call would result in an immediate risk-free profit. Moreover, we show $C_E(S_t, t) \leq S_t$. Assume $C_E(S_t, t) > S_t$. Buy the asset, sell the call and eventually sell the asset on maturity date $T$. An immediate risk-free profit $C_E(S_t, t) - S_t > 0$ is realized contradicting the no-arbitrage principle.

For the proof of the lower bound in $(\ast)$ assume the existence of an $0 \leq t^* \leq T$ such that

$$C_E(S_{t^*}, t^*) < S_{t^*} - K \exp(-r(T - t^*)).$$

The following arbitrage table shows that a risk-free profit is realized at time $t^*$ contradicting the no-arbitrage principle.
### Arbitrage Table for the Proof of Theorem 1.2

Arbitrage table for the proof of the lower bound in (*)

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Cash Flow</th>
<th>Value Portfolio at t</th>
<th>Value Portfolio at T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sell $S_{t^*}$</td>
<td>$S_{t^*}$</td>
<td>$-S_{t^*}$</td>
<td>$-S_T$ $-S_T$</td>
</tr>
<tr>
<td>Buy $C_E(S_{t^<em>}, t^</em>)$</td>
<td>$-C_E(S_{t^<em>}, t^</em>)$</td>
<td>$C_E(S_{t^<em>}, t^</em>)$</td>
<td>$0$ $S_T - K$</td>
</tr>
<tr>
<td>Invest $K \exp(-r(T - t^*))$</td>
<td>$-K \exp(-r(T - t^*))$</td>
<td>$K \exp(-r(T - t^*))$</td>
<td>$K$ $K$</td>
</tr>
<tr>
<td>Sum</td>
<td>$\pi_{t^<em>} - K \exp(-r(T - t^</em>)) &gt; 0$</td>
<td>$K \exp(-r(T - t^<em>)) - \pi_{t^</em>} &lt; 0$</td>
<td>$K - S_T \geq 0$ $0$</td>
</tr>
</tbody>
</table>

The proof of (**) is left as an exercise.
Theorem 1.3 Lower and Upper Bounds for American Options

Let $K, S_t, P_A(S_t, t)$ and $C_A(S_t, t)$ be the values of a bond (with interest rate $r > 0$ and proportional yield), an asset, an American put, and an American call. Under the previous assumptions, for $0 \leq t \leq T$ there holds

1. \[ C_A(S_t, t) = C_E(S_t, t), \]
2. \[ K \exp(-r(T-t)) \leq S_t + P_A(S_t, t) - C_A(S_t, t) \leq K, \]
3. \[ (K \exp(-r(T-t)) - S_t)^+ \leq P_A(S_t, t) \leq K. \]

Proof of (1). Assume that the American call is exercised at time $t < T$ which, of course, only makes sense when $S_t > K$. On the other hand, according to Theorem 1.2 (*), which also holds true for American options, we have

\[ C_A(S_t, t) \geq (S_t - K \exp(-r(T-t)))^+ = S_t - K \exp(-r(T-t)) > S_t - K, \]

i.e., it is preferable to sell the option instead of exercising it. Hence, the early exercise is not optimal. But exercising on maturity date $T$ corresponds to the case of a European option.
Proof of (++): Obviously, the higher flexibility of American put options implies that $P_A(S_t, t) \geq P_E(S_t, t), 0 \leq t \leq T$. Then, (+) and the put-call parity Theorem 1.1 yield

$$C_A(S_t, t) - P_A(S_t, t) \leq C_E(S_t, t) - P_E(S_t, t) = S_t - K \exp(-r(T - t)),$$

which is the lower bound in (++)

The upper bound is verified by the following arbitrage table:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Cash Flow</th>
<th>Value Portfolio at $t$</th>
<th>Value Portfolio at $T^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$S_{T^*} \leq K$</td>
<td>$S_{T^*} &gt; K$</td>
</tr>
<tr>
<td>Sell put</td>
<td>$P_A(S_t, t)$</td>
<td>$-P_A(S_t, t)$</td>
<td>$-(K - S_{T^*})$</td>
</tr>
<tr>
<td>Buy call</td>
<td>$-C_A(S_t, t)$</td>
<td>$C_A(S_t, t)$</td>
<td>$\geq 0$</td>
</tr>
<tr>
<td>Sell asset</td>
<td>$S_t$</td>
<td>$-S_t$</td>
<td>$-S_{T^*}$</td>
</tr>
<tr>
<td>Invest K</td>
<td>$-K$</td>
<td>$K$</td>
<td>$K \exp(r(T^* - t))$</td>
</tr>
<tr>
<td>Sum</td>
<td>$P_A - C_A$</td>
<td>$-P_A + C_A$</td>
<td>$\geq K (\exp(r(T^* - t)) - 1)$</td>
</tr>
<tr>
<td></td>
<td>$+S - K &gt; 0$</td>
<td>$-S + K &lt; 0$</td>
<td>$\geq 0$</td>
</tr>
</tbody>
</table>
Proof of (+++). We note that the chain (+++) of inequalities can be equivalently stated as

\[ K \exp(-r(T-t)) - S_t + C_A(S_t, t) \leq P_A(S_t, t) \leq K - S_t + C_A(S_t, t) . \]

Using (+) and the lower bound in Theorem 1.2 (*), we find

\[ P_A(S_t, t) \geq K \exp(-r(T-t)) - S_t + C_A(S_t, t) = K \exp(-r(T-t)) - S_t + C_E(S_t, t) \geq \]
\[ \geq K \exp(-r(T-t)) - S_t + (S_t - K \exp(-r(T-t)))^+ = (K \exp(-r(T-t)) - S_t)^+ . \]

On the other hand, using again (+) and the upper bound in Theorem 1.2 (*) yields

\[ P_A(S_t, t) \leq K - S_t + C_A(S_t, t) = K - S_t + C_E(S_t, t) \leq \]
\[ \leq K - S_t + S_t = K , \]

which proves (+++).
Lower and Upper Bounds for European and American Options

\[ K e^{-r(T-t)} \]

\[ K \]

\[ P_A(t) \]

\[ P_E(t) \]

\[ S_t \]

\[ C_A(t) = C_E(t) \]

\[ K e^{-r(T-t)} \]

Puts

Calls