Chapter 2: Binomial Methods and the Black-Scholes Formula

2.1 Binomial Trees

We consider a financial market consisting of a bond $B_t = B(t)$, a stock $S_t = S(t)$, and a call-option $C_t = C(t)$, where the trade is only possible at time $t = 0$ and $t = \Delta t$.

Assumptions:

- There is a fixed interest rate $r > 0$ on the bond with initial value $B_0 = 1$. Taking proportional yield into account, at $t = \Delta t$ there holds $B_{\Delta t} = \exp(r\Delta t)$.
- There are only two possibilities for the price $S_{\Delta t}$ of the stock with initial value $S = S_0$ at time $t = \Delta t$:
  - Either $S_{\Delta t} = u \cdot S$ (up) with probability $P(\text{up}) = q$, $0 < q < 1$, or $S_{\Delta t} = d \cdot S$ (down) with probability $P(\text{down}) = 1 - q$, where $u > d > 0$.
  - The price of the call-option is $K$ and the maturity date is $T$.
  - There is no-arbitrage and short sellings are allowed (i.e., selling stocks that are not yet owned but delivered later). There are no transaction costs and no dividends on the stocks.
Lemma 2.1  The no-arbitrage principle and the possibility of short sellings imply

\[ d \leq \exp(r\Delta t) \leq u. \]

Proof. Assume \( \exp(r\Delta t) > u \). Then, the purchase of a bond by short sellings results in an immediate profit.

On the other hand, if \( \exp(r\Delta t) < d \), a risk-free profit can be realized by the purchase of the stock financed by a credit.

Both cases contradict the no-arbitrage principle.
Price of a European Call-Option in the One-Period Model

Value of the call-option at time $t = \Delta t$:

$$
(+)
(\text{Up-State}) \quad C_u := (uS - K)^+ \quad , \quad (\text{Down-State}) \quad C_d := (dS - K)^+. 
$$

Computation of the price $C_0$ by the duplication strategy:

Buy resp. sell $c_1$ bond and $c_2$ stock such that

$$
(\circ)_1 \quad c_1 \cdot B_0 + c_2 \cdot S_0 = C_0 ,
(\circ)_2 \quad c_1 \cdot B_{\Delta t} + c_2 \cdot S_{\Delta t} = C_{\Delta t} .
$$

Using $(+)$ in $(\circ)_2$, we obtain the following linear system in $c_1, c_2$:

$$
C_1 \exp(r\Delta t) + C_2 uS = C_u ,
C_1 \exp(r\Delta t) + C_2 dS = C_d ,
$$

whose solution is given by

$$
(*) \quad c_1 = \frac{uC_d - dC_u}{(u - d) \exp(r\Delta t)} , \quad c_2 = \frac{C_u - C_d}{(u - d) S} .
$$
Price of a European Call-Option in the One-Period Model

Inserting (*) into (◦)\textsubscript{1} results in

\begin{equation}
C_0 = \exp(-r\Delta t) \left( p C_u + (1 - p) C_p \right), \quad p = \frac{\exp(r\Delta t) - d}{u - d}.
\end{equation}

Interpretation of the option price as a discounted expectation

Since \( d \leq \exp(r\Delta t) \leq u \) (cf. Lemma 2.1), we have \( 0 \leq p \leq 1 \). Recalling that the expectation of a random variable \( X \) attaining the states \( X_u \) resp. \( X_d \) with probability \( p \) resp. \( 1 - p \) is given by

\[ E_p(X) = p \, X_u + (1 - p) \, X_d, \]

from (†) we deduce

\[ C_0 = \exp(-r\Delta t) \, E_p((S_{\Delta t} - K)^+) \, . \]

In view of

\[ E_p(S_{\Delta t}) = p \, u \, S + (1 - p) \, d \, S = \exp(r\Delta t) \, S, \]

\( p \) can be interpreted as a risk-neutral probability (the expected value of the asset with probability \( p \) of the up-state equals the profit from the risk-free bond).
Price of a European Call-Option in the n-Period Model (Cox-Ross-Rubinstein Model)

Under the same assumptions as before, we consider an n-period model of the financial market where for each time interval of length $\Delta t$ the value of the stock may change by the factor $u$ with probability $q$ and by factor $d$ with probability $1-q$. Hence, assuming $k$ up-states and $n-k$ down-states, the value of the stock at maturity date $T = n\Delta t, n \in \mathbb{N}$, is given by

$$S^n_k := u^k d^{n-k} S.$$ 

**Theorem 2.2 (Cox-Ross-Rubinstein Model)**

The price $C_0$ of a European call-option in the n-period model is

$$C_0 = \exp(-rn\Delta t) \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (S^n_k - K)^+.$$ 

**Proof.** The proof is by induction (Exercise).
Discrete Black-Scholes Formula

We may interpret \( \binom{n}{k} p^k (1-p)^{n-k} \) as the probability that the stock attains the value \( S^k \) at time \( T = n\Delta t \) and

\[
E_p(X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} X_k
\]

as the expectation of a random variable \( X \) which attains the state \( X_k, 0 \leq k \leq n \), with probability \( \binom{n}{k} p^k (1-p)^{n-k} \). Hence, the option price \( C_0 \) can be written as the discounted expectation

\[
\text{(‡)} \quad C_0 = \exp(-rT) E_p((S^n - K)^+) .
\]

**Theorem 2.3 (Discrete Black-Scholes Formula)**

With \( m := \min \{0 \leq k \leq n \mid u^k d^{n-k} S - K \geq 0\} \) and \( p' := pu \exp(-r\Delta t) \) there holds

\[
C_0 = S \Phi(m, p') - K \exp(-rT) \Phi(m, p) , \quad \Phi(m, p) = \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k} .
\]

**Proof.** The proof follows readily from (‡) observing \( 1-p' = (1-p) d \exp(-r\Delta t) \).
2.2 A Stochastic Model for the Value of a Stock

Definition 2.1 (Wiener Process, Brownian Motion)

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, i.e., \(\Omega\) is a set, \(\mathcal{F} \subset \mathcal{P}(\Omega)\) is a \(\sigma\)-algebra with \(\mathcal{P}(\Omega)\) being the power set of \(\Omega\), and \(\mathbb{P} : \mathcal{F} \rightarrow [0, 1]\) is a probability measure on \(\mathcal{F}\).

A Wiener process or Brownian motion is a continuous stochastic process \(W_t = W(\cdot, t)\) where \(W : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}\) with the properties

\((W_1)\) \(W_0 = 0\) almost sure, i.e., \(\mathbb{P}(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1\).

\((W_2)\) \(W_t \sim \mathcal{N}(0, t)\), i.e., \(W_t\) is \(\mathcal{N}(0, t)\)-distributed. This means that for \(t \in \mathbb{R}_+\) the random variable \(W_t\) is normally distributed with mean \(\mathbb{E}(W_t) = 0\) and variance \(\text{Var}(W_t) = \mathbb{E}(W_t^2) = t\).

\((W_3)\) All increments \(\Delta W_t := W_{t+\Delta t} - W_t\) on non-overlapping time intervals are independent, i.e., \(W_{t_2} - W_{t_1}\) and \(W_{t_4} - W_{t_3}\) are independent for all \(0 \leq t_1 < t_2 \leq t_3 < t_4\).

Theorem 2.4 (Properties of a Wiener Process)

A Wiener process \(W_t\) has the properties that for all \(0 \leq s < t\) there holds

\[(i)\] \(\mathbb{E}(W_t - W_s) = 0\) \quad \[(ii)\] \(\text{Var}(W_t - W_s) = \mathbb{E}((W_t - W_s)^2) = t - s\).
A Discrete-Time Model of a Wiener Process

For the discrete times \( t_m := m\Delta t, m \in \mathbb{N} \), where \( \Delta t > 0 \), the value \( W_t \) of a Wiener process can be written as the sum of independent and normally distributed increments \( \Delta W_k \) according to

\[
W_{m\Delta t} = \sum_{k=1}^{m} \left( W_{k\Delta t} - W_{(k-1)\Delta t} \right) =: \Delta W_k
\]

Increments \( \Delta W_k \) with such a distribution and \( \text{Var}(\Delta W_k) = \Delta t \) can be computed from standard normally distributed random numbers \( Z \), i.e.,

\[
Z \sim N(0,1) \implies Z \cdot \sqrt{\Delta t} \sim N(0, \Delta t).
\]

This gives rise to the following discrete model of a Wiener process

\[
\Delta W_k = Z \sqrt{\Delta t}, \quad \text{where } Z \sim N(0,1).
\]

Remark: The computation of \( Z \) will be explained in Chapter 4.
Discrete-Time Model of a Wiener Process with $\Delta t = 0.0002$

Realization of a Wiener process; courtesy of [Günter/Jüngel]
Dow Jones Index at 500 trading days from Sept. 8, 1997 to August 31, 1999
A Stochastic Model for the Value of a Stock

Idea: Consider a bond $B_t$ with risk-free interest rate $r > 0$ and proportional yield. Then, there holds $B_t = B_0 \exp(rt)$ which is equivalent to

$$\ln B_t = \ln B_0 + r \cdot t.$$ 

Taking into account the uncertainty of the stock market, for the value $S_t$ of the stock we assume

$$\ln S_t = \ln S_0 + b \cdot t + \text{'uncertainty'}. $$

As far as the uncertainty is concerned, we assume that it has expectation 0 and is $N(0, \sigma^2 t)$-distributed which, in view of $\text{Var}(\sigma W_t) = \sigma^2 t$, suggests

$$\text{(⊙)} \quad \ln S_t = \ln S_0 + b \cdot t + \sigma W_t. $$

Definition 2.2 (Geometric Brownian Motion) Setting $\mu := b + \sigma^2/2$, we deduce from (⊙)

$$\text{(⋆)} \quad S_t = S_0 \exp(\mu t + \sigma W_t - \frac{1}{2} \sigma^2 t).$$

$S_t$ is called a geometric Brownian motion. Note that $S_t$ is log-normally distributed.
Lemma 2.2 Properties of geometric Brownian motions
For the geometric Brownian motion $S_t$ a given in Definition 2.2 there holds

(i) $E(S_t) = S_0 \exp(\mu t)$,

(ii) $\text{Var}(S_t) = S_0^2 \exp(2\mu t) (\exp(\sigma^2 t) - 1)$.

Proof. Since $W_t$ is $N(0,t)$-distributed, we have

$$E(\exp(\sigma W_t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp(\sigma x) \exp(-x^2/2t) \, dx =$$

$$= \frac{1}{\sqrt{2\pi t}} \exp(\sigma^2 t/2) \int_{\mathbb{R}} \exp(-(x - \sigma t)^2/2t) \, dx = \exp(\sigma^2 t/2),$$

whence

$$E(S_t) = S_0 \exp(\mu t - \sigma^2 t/2) E(\exp(\sigma W_t)) = S_0 \exp(\mu t).$$

Moreover, we obtain

$$\text{Var}(S_t) = E(S_t^2) - E(S_t)^2 = S_0^2 \exp((2\mu - \sigma^2) t) E(\exp(2\sigma W_t)) - S_0^2 \exp(2\mu t) =$$

$$= S_0^2 \exp(2\mu t) (\exp(\sigma^2 t) - 1).$$
2.3 The Continuous Black-Scholes Formula

We recall that in an n-period model the price of a European call-option is given by

\[ C_0 = S \mathbb{P}(X_{p'} \geq m) - K \exp(-rT) \mathbb{P}(X_p \geq m), \]

where \( m = \min\{0 \leq k \leq n \mid u^kd^n-kS - K \geq 0\} \) and \( X_{p'} \) resp. \( X_p \) are \( B(n,p') \) resp. \( B(n,p) \)-distributed random variables with

\[ p = \frac{\exp(r\Delta t) - d}{u - d}, \quad p' = p \, u \exp(-r\Delta t). \]

**Theorem 2.5 (Continuous Black-Scholes Formula)**

Assume that \( T = n \Delta t \), \( u > 1 \), \( d = 1/u \) and define \( \sigma > 0 \) such that \( u = \exp(\sigma \sqrt{\Delta t}) \) and \( d = \exp(-\sigma \sqrt{\Delta t}) \). Then, there holds

\[ \lim_{\Delta t \to 0} C_0 = S \Phi(d_1) - K \exp(-rT) \Phi(d_2), \quad \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-s^2/2) \, ds, \]

where \( d_1, d_2 \) are given by

\[ d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}}. \]
Proof. It is sufficient to verify\[\lim_{\Delta t \to 0} P(X_p \geq m) = \Phi(d_2) , \quad \lim_{\Delta t \to 0} P(X_p' \geq m) = \Phi(d_1) .\]

We prove the first assertion and leave the second one as an exercise.
To this end, we reformulate \(P(X_p \geq m)\) according to\[\begin{align*}
\oplus \quad P(X_p \geq m) = 1 - P(X_p < m) = 1 - P\left(\frac{X_p - np}{\sqrt{np(1-p)}} < \frac{m - np}{\sqrt{np(1-p)}}\right).
\end{align*}\]

In view of the definition of \(m\), we have
\[m \ln u + (n - m) \ln d \geq \ln \frac{K}{S} \iff m \geq -\frac{\ln(S/K) + n \ln d}{\ln(u/d)} .\]

We choose \(0 \leq \alpha < 1\) such that\[\begin{align*}
\odot \quad m = -\frac{\ln(S/K) + n \ln d}{\ln(u/d)} + \alpha .
\end{align*}\]

Inserting \((\odot)\) into \((\oplus)\) gives\[\begin{align*}
\otimes \quad P(X_p \geq m) = 1 - P\left(\frac{X_p - np}{\sqrt{np(1-p)}} < \frac{-\ln(S/K) - n(p \ln(u/d) + \ln d) + \alpha \ln(u/d)}{\ln(u/d) \sqrt{np(1-p)}}\right).
\end{align*}\]
We apply the **central limit theorem** for $B(n, p)$-distributed random variables to $(\otimes)$.

**Theorem 2.6 (Central Limit Theorem for $B(n, p)$-Distributed Random Variables)**

For a sequence $(Y_n)_{n \in \mathbb{N}}$ of $B(n, p)$-distributed random variables in a probability space there holds

\[
\lim_{n \to \infty} P\left( \frac{Y_n - np}{\sqrt{np(1-p)}} \leq x \right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-s^2/2) \, ds .
\]

Continuation of the proof of Thm 3.5. In order to apply $(\ast)$, we have to evaluate the limits

\[
\lim_{\Delta t \to 0} n \, p \, (1 - p) \left( \ln \frac{u}{d} \right)^2 , \quad \lim_{\Delta t \to 0} n \, (p \ln \frac{u}{d} + \ln d) .
\]

Taylor expansion of $p$ as a function of $\Delta t$ around 0 yields

\[
p = \frac{\exp(r\Delta t) - \exp(-\sigma\sqrt{\Delta t})}{\exp(\sigma\sqrt{\Delta t}) - \exp(-\sigma\sqrt{\Delta t})} = \frac{\sigma + (r - \sigma^2/2) \sqrt{\Delta t} + O(\Delta t)}{2\sigma + O(\Delta t)} ,
\]

whence

\[
(\circ) \quad \lim_{\Delta t \to 0} p = \frac{1}{2} , \quad \lim_{\Delta t \to 0} \frac{2p - 1}{\sqrt{\Delta t}} = \frac{r}{\sigma} - \frac{\sigma}{2} .
\]
An immediate consequence of (o) is

\[
\lim_{\Delta t \to 0} n \ p \ (1 - p) \ (\ln \frac{u}{d})^2 = \lim_{\Delta t \to 0} \frac{T}{\Delta t} \ p (1 - p) (2\sigma\sqrt{\Delta t})^2 \ = \lim_{\Delta t \to 0} 4p(1 - p) \ \sigma^2 \ T = \sigma^2 \ T ,
\]

\[
\lim_{\Delta t \to 0} n \ (p \ ln \frac{u}{d} + \ln d) = \lim_{\Delta t \to 0} \frac{T}{\sqrt{\Delta t}}(2p - 1) \ \sigma = (r - \frac{\sigma^2}{2}) \ T .
\]

Now, the application of the central limit theorem (Theorem 2.6) results in

\[
P(X_p \geq m) \to 1 - \Phi\left(\frac{-\ln(S/K) - (r - \sigma^2/2) \ T}{\sigma \sqrt{T}}\right).
\]

Observing \(1 - \Phi(-x) = \Phi(x)\) finally allows to conclude:

\[
P(X_p \geq m) \to \Phi\left(\frac{\ln(S/K) + (r - \sigma^2/2) \ T}{\sigma \sqrt{T}}\right) = \Phi(d_2) .
\]
2.4 The Binomial Method

The binomial method provides an algorithmic tool for the computation of an approximation of the price of a European or an American option. We partition the time interval \([0, T]\) into \(N\) equidistant subintervals of length \(\Delta t = T/N, N \in \mathbb{N}\), and compute approximations \(S_{t_i} = S_{T_i}, 0 \leq i \leq N\), at times \(t_i = i \Delta t\).

We make the following assumptions:

- The value of the stock at time \(t_{i+1}\) is either \(S_{i+1} = u S_i\) with probability \(p \in (0, 1)\) (‘up’) or it is \(S_{i+1} = d S_i\) with probability \(1 - p\) (‘down’).
- The expected profit within \(\Delta t\) corresponds to the risk-free interest, i.e., with \(\mu = r\) we obtain

\[
E(S(t_{i+1})) = S(t_i) \exp(r \Delta t) , \quad \text{Var}(S(t_{i+1})) = S(t_i)^2 \exp(2r \Delta t) (\exp(\sigma^2 \Delta t) - 1) .
\]

Likewise, for the option price \(V(t_i)\) we assume

\[
E(V(t_{i+1})) = V(t_i) \exp(r \Delta t) .
\]

- There are no transaction costs and there are no dividends on the stocks.
Specification of the parameters $u, d, p$

The three parameters $u, d$ and $p$ in the binomial method can be determined by a nonlinear system of three equations. Two of these equations can be obtained by assuming that the expectation and variance of the value of the stock at $t_{i+1}$ coincide for the time-continuous model and the time-discrete model. For the time-discrete model, we have

\[
\begin{align*}
\mathbb{E}(S_{i+1}) &= p \cdot u S_i + (1 - p) \cdot d S_i, \\
\text{Var}(S_{i+1}) &= p (uS_i)^2 + (1 - p)(dS_i)^2 - (puS_i + (1 - p)dS_i)^2.
\end{align*}
\]

Replacing $S(T_i)$ by $S_i$ in $(\ast)$ yields

\[
\begin{align*}
\text{(i)} & \quad S_i \exp(r\Delta t) = p \cdot u S_i + (1 - p) \cdot d S_i, \\
\text{(ii)} & \quad S_i^2 \exp(2r\Delta t) (\exp(\sigma^2\Delta t) - 1) = p (uS_i)^2 + (1 - p)(dS_i)^2 - (puS_i + (1 - p)dS_i)^2.
\end{align*}
\]

The two equations $(\text{i}), (\text{ii})$ have to be complemented by a third one. There are two options:

- **Variant I:** $(\text{iii}) \quad u \cdot d = 1$ (symmetry w.r.t. ’up’ and ’down’)
- **Variant II:** $(\text{iii}) \quad p = \frac{1}{2}$ (same probability for ’up’ and ’down’)


Variant I: The solution of $(\circ), (\circ\circ), (\circ\circ\circ)$ is given by
\[
\begin{align*}
    u &= \beta + \sqrt{\beta^2 - 1}, \\
    d &= \beta - \sqrt{\beta^2 - 1}, \\
    p &= \frac{\exp(r\Delta t) - d}{u - d}, \quad \beta = \frac{1}{2} \left( \exp(-r\Delta t) + \exp((r + \sigma^2)\Delta t) \right).
\end{align*}
\]

Variant II: In this case, the solution of $(\circ), (\circ\circ), (\circ\circ\circ)$ turns out to be
\[
\begin{align*}
    u &= \exp(r\Delta t) \left( 1 + \sqrt{\exp(\sigma^2\Delta t) - 1} \right), \\
    d &= \exp(r\Delta t) \left( 1 - \sqrt{\exp(\sigma^2\Delta t) - 1} \right), \\
    p &= \frac{1}{2}.
\end{align*}
\]
**Algorithm 2.1: Binomial Method**

Denoting by \( S_0 \) the value of the stock at \( t = 0 \) and setting \( S_{ji} := u^j d^i S_0 \), \( 0 \leq i \leq N, 0 \leq j \leq i \), we proceed as follows

**Step 1:** Initialization of the binomial tree

For \( j = 0, 1, \ldots, N \) compute

\[
S_{jN} = u^j d^{N-j} S_0 .
\]

**Step 2:** Computation of the option prices

For \( j = 0, 1, \ldots, N \) compute

\[
V_{jN} = \begin{cases} (S_{jN} - k)^+, & \text{Call} \\ (K - S_{jN})^+, & \text{Put} \end{cases}
\]

**Step 3:** Backward Iteration

We remark that in terms of \( S_{ji} \), the first equation \((\circ)\) can be written as

\[
S_{ji} \exp(r\Delta t) = p u S_{ji} + (1 - p) d S_{ji} = p S_{j+1, i+1} + (1 - p) S_{j, i+1} .
\]
Step 3: Backward Iteration (Continuation)

If we replace the option price $V(t_i)$ in $V(t)$ by its discrete counterpart $V_i$, we obtain

$$V_{ji} \exp(r\Delta t) = p \, V_{j+1,i+1} + (1 - p) \, V_{j,i+1}.$$

Consequently, the backward iteration is implemented as follows:

For $i = N - 1, N - 2, ..., 0$ and $j = 0, 1, ..., i$ compute

$$V_{ji} = \exp(-r\Delta t) \,(p \, V_{j+1,i+1} + (1 - p) \, V_{j,i+1})$$

in case of European option and

$$\tilde{V}_{ji} = \exp(-r\Delta t) \,(p \, V_{j+1,i+1} + (1 - p) \, V_{j,i+1}),$$

$$V_{ji} = \begin{cases} \max\{(S_{ji} - K)^+, \tilde{V}_{ji}\}, & \text{Call} \\ \max\{(K - S_{ji})^+, \tilde{V}_{ji}\}, & \text{Put} \end{cases}$$

for an American option.
2.6 Implementation of the Binomial Method in MATLAB

The MATLAB program binbaum1.m computes the price of an European put option according to the binomial method.

The input parameters have to be specified by the user.

The commands will be sequentially compiled and executed by the MATLAB interpreter.

For appropriate outputs see the MATLAB handbook.

```matlab
% Input parameters
K = 0; S0 = 0; r = 0; sigma = 0; T = 0; N = 0;
% Computation of u, d, p
beta = 0.5 * (exp(-r * dt) + exp((r + sigma^2) * dt));
u = beta + sqrt(beta^2 - 1);
d = 1/u;
p = (exp(r * dt) - d) / (u - d);

% First step
for j = 1 : N + 1
    S(j, N + 1) = S0 * u^j * d^(N - j + 1)
end

% Second step
for j = 1 : N + 1
    V(j, N + 1) = max(K - S(j, N + 1), 0);
end

% Third step
for i = N : -1 : 1
    for j = 1 : i
        V(j, i) = e * (p * V(j + 1, i + 1) + (1 - p) * V(j, i + 1));
    end
end

% Output
fprintf('V(%f, 0) = %f \n', S0, V(1, 1))
```
Vectorized MATLAB program binbaum2.m

The MATLAB program **binbaum2.m** computes the price of an European put option by a vectorized version of the binomial method. For large $N$, the vectorized version is by orders of magnitude faster than the standard version **binbaum1.m**.

```matlab
function V = binbaum2(S0, K, r, sigma, T, N)
    % Computation of u, d, p
    dt = T/N;
    beta = 0.5 * (exp(-r * dt) + exp((r + sigma^2) * dt));
    u = beta + sqrt(beta^2 - 1);
    d = 1/u;
    p = (exp(r * dt) - d)/(u - d);
    % First step
    S = S0 .* (u.^((0:N)') .* d.^((N:-1:0)'));
    % Second step
    V = max(K - S, 0);
    end
    % Third step
    q = 1 - p;
    for i = N:-1:1
        V = p * V(2:i+1) + q * V(1:i);
    end
    % Output
    V = exp(-r * T) * V;
```