Chapter 3: Black-Scholes Equation and Its Numerical Evaluation

3.1 Itô Integral

3.1.1 Convergence in the Mean and Stieltjes Integral

Definition 3.1 (Convergence in the Mean)
A sequence \( \{X_n\}_{n \in \mathbb{N}} \) of random variables is said to converge in the mean to a random variable \( X \), if \( \mathbb{E}(X_n^2) < \infty \), \( \mathbb{E}(X^2) < \infty \) and if \( \lim_{n \to \infty} \mathbb{E}((X - X_n)^2) = 0 \). We use the notation \( \text{l.i.m.}_{n \to \infty} X_n = X \).

Definition 3.2 (Stieltjes Integral)
Let \( b \in C([0, T]) \) and assume that \( \mu \in \text{BV}([0, T]) \) is a function of bounded variation, i.e., for a sequence \( 0 = t_0^{(N)} < t_1^{(N)} < ... < t_N^{(N)} = T \) of partitions of \([0, T]\) with \( \max_{1 \leq j \leq N} |t_j^{(N)} - t_{j-1}^{(N)}| \to 0 \) as \( N \to \infty \)

\[
\lim_{N \to \infty} \sum_{j=1}^{N} |\mu(t_j) - \mu(t_{j-1})| < \infty.
\]

The Stieltjes integral of \( b \) with respect to \( \mu \) is defined according to

\[
\int_0^T b(t) \, d\mu(t) = \lim_{N \to \infty} \sum_{j=1}^{N} b(t_{j-1}) (\mu(t_j) - \mu(t_{j-1})).
\]
3.1.2 First and Second Variation of a Wiener Process

Adopting the notation from the previous subsection, for a Wiener process $W_t$ the mapping $t \mapsto W_t$ is not of bounded variation, i.e., for the first variation of $W_t$ we have

$$\lim_{N \to \infty} \sum_{j=1}^{N} |W_{t_{j}^{(N)}} - W_{t_{j-1}^{(N)}}| = \infty .$$

However, using $E((W_t - W_s)^2) = t - s$, for the second variation of $W_t$ we find

$$\sum_{j=1}^{N} E((W_{t_{j-1}^{(N)}} - W_{t_{j}^{(N)}})^2) = \sum_{j=1}^{N} (t_{j}^{(N)} - t_{j-1}^{(N)}) = t_{N}^{(N)} - t_{0}^{(N)} = T .$$

whence

$$\lim_{N \to \infty} \sum_{j=1}^{N} (W_{t_{j}^{(N)}} - W_{t_{j-1}^{(N)}})^2 = T .$$
3.1.3 Motivation of the Itô Integral

Assume that the temporal evolution of an asset occurs according to a Wiener process $W_t$ and denote by the function $b = b(t)$ the number of units of the asset in a portfolio at time $t$.

(i) Discrete time trading

If trading is only allowed at discrete times $t_j$, given by a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $[0, T]$, the trading gain turns out to be

$$\sum_{j=1}^{N} b(t_{j-1}) (W_{t_j} - W_{t_{j-1}}).$$

Definition 3.3 (Itô Integral over a Step Function)

Assume that $b$ is a step function $b(t) = b(t_{j-1})$, $t_{j-1} \leq t < t_j$, $1 \leq j \leq N$. The Itô integral of $b$ is defined by means of

$$\int_0^T b(t) \, dW_t = \sum_{j=1}^{N} b(t_{j-1}) (W_{t_j} - W_{t_{j-1}}).$$

In case the values $b(t_{j-1})$ are random variables, $b$ is called a simple process.
(ii) Continuous time trading

If trading is allowed at all times $t \in [0, T]$ and $b \in C([0, T])$, one might be tempted to define the continuous time trading gain as the limit $N \to \infty$ of its discrete counterpart. However, this limit does not exist, since the Wiener process $W_t$ has an unbounded first variation.

A remedy is to approximate $b$ by step functions $b_n, n \in \mathbb{N}$, in the sense that

$$\lim_{n \to \infty} E\left(\int_0^T (b(t) - b_n(t))^2 \, dt\right) = 0.$$

In view of the isometry

$$E(\int_0^T (b_n(t) - b_m(t)) \, dW_t)^2 = E(\int_0^T (b_n(t) - b_m(t))^2 \, dt),$$

and the Cauchy convergence

$$E(\int_0^T (b_n(t) - b_m(t))^2 \, dt) \to 0,$$

we deduce that the Itô integrals $\int_0^T b_n(t) \, dW_t$ represent a Cauchy sequence with respect to the convergence in the mean.
3.1.4 Itô Integral

Definition 3.4 (Itô Integral)
Assume that $b = b(t)$ is a stochastically integrable function in the sense that there exists a sequence $\{b_n\}_{n \in \mathbb{N}}$ of simple processes such that

$$\lim_{n \to \infty} E\left( \int_0^T (b(t) - b_n(t))^2 \, dt \right) = 0.$$ 

Then, the Itô integral of $b$ is defined according to

$$\int_0^T b(t) \, dW_t := \lim_{n \to \infty} \int_0^T b_n(t) \, dW_t.$$
3.2 Stochastic Differential Equations and Itô Processes

Definition 3.5 (Stochastic Differential Equation)
Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(X_t, t \in \mathbb{R}_+\) be a stochastic process \(X : \Omega \times \mathbb{R}_+ \to \mathbb{R}\). Moreover, assume that \(a(\cdot, \cdot) : \Omega \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}\) and \(b(\cdot, \cdot) : \Omega \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}\) are stochastically integrable functions of \(t \in \mathbb{R}_+\). Then, the equation
\[
\text{d}X_t = a(X_t, t) \text{d}t + b(X_t, t) \text{d}W_t
\]
is called a stochastic differential equation. Note that (\(*\)) has to be understood as a symbolic notation of the stochastic integral equation
\[
\begin{align*}
X_t &= X_0 + \int_0^t a(X_s, s) \text{d}s + \int_0^t b(X_s, s) \text{d}W_s .
\end{align*}
\]
The functions \(a\) and \(b\) are referred to as the drift term and the diffusion term, respectively.

Definition 3.6 (Itô Process)
A stochastic process \(X_t\) satisfying (\(*\)) is said to be an Itô process.
Examples: (i) Wiener process as a special Itô process
Setting \( a \equiv 0 \) and \( b \equiv 1 \) in \((\ast\ast)\), we obtain
\[
X_t = X_0 + \int_0^t dW_s = X_0 + W_t - W_0 = X_0 + W_t.
\]

(ii) Stochastic differential equation for a bond with risk-free interest rate \( r \in \mathbb{R}_+ \)
In case \( a(X_t, t) = rX_t, r \in \mathbb{R}_+ \) and \( b \equiv 0 \), we get
\[
X_t = X_0 + r \int_0^t X_s \, ds \quad \iff \quad dX_t = rX_t \, dt.
\]
This is a stochastic differential equation for a bond \( X_t \) with risk-free interest rate \( r \in \mathbb{R}_+ \).

(iii) Itô integral of a Wiener process
Setting \( X_t = W_t \), \( \tilde{W}_t := W_{t_j-1}, t_{j-1} \leq t < t_j \), and observing \( \lim_{N \to \infty} \sum_{j=1}^N (W_{t_j} - W_{t_j-1})^2 = T \), we get
\[
\int_0^T W_t \, dW_t = \lim_{N \to \infty} \int_0^T \tilde{W}_t \, dW_t = \lim_{N \to \infty} \sum_{j=1}^N W_{t_{j-1}} (W_{t_j} - W_{t_{j-1}}) = \frac{1}{2} W_T^2 - \frac{1}{2} \lim_{N \to \infty} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})^2 = \frac{1}{2} W_T^2 - \frac{T}{2}. \]
3.2 Itô’s Lemma

Lemma 3.1 (Itô’s Lemma)

Let $X_t, t \in \mathbb{R}_+$, be an Itô process $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $f : C^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$. Then, the stochastic process $f_t := f(X_t, t)$ is also an Itô process which satisfies

$$df_t = \left( \frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} \right) dt + b \frac{\partial f}{\partial x} dW_t .$$

Proof. Taylor expansion of $f(X_{t+\Delta t}, t + \Delta t)$ around $(X_t, t)$ results in

$$f(X_{t+\Delta t}, t + \Delta t) = f(X_t, t) + \frac{\partial f}{\partial t}(X_t, t) \Delta t + \frac{\partial f}{\partial x}(X_t, t) (X_{t+\Delta t} - X_t) +$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial t^2}(X_t, t) (\Delta t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(X_t, t) (X_{t+\Delta t} - X_t)^2 + \frac{\partial^2 f}{\partial x \partial t}(X_t, t) \Delta t (X_{t+\Delta t} - X_t) +$$

$$+ O((\Delta t)^2) + O((\Delta t)(X_{t+\Delta t} - X_t)^2) + O((X_{t+\Delta t} - X_t)^3) .$$
Passing to the limit $\Delta t \to 0$ yields

$$\begin{array}{r}
\text{(+) } df_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dX_t^2 + O((dt)^2) + O(dt (dX_t)^2) + O((dX_t)^3) .
\end{array}$$

Taking into account that $X_t$ is an Itô process and $dW_t^2 = dt$, it follows that

$$\begin{array}{r}
\text{(+++) } dX_t^2 = (a dt + b dW_t)^2 = a^2 (dt)^2 + 2 a b dt dW_t + b^2 dW_t^2 = b^2 dt + O((dt)^{3/2}) .
\end{array}$$

Using (+++) in (+), we finally obtain

$$\begin{array}{r}
df_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (a dt + b dW_t) + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} dt = \\
\quad = \left( \frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + b \frac{\partial f}{\partial x} dW_t .
\end{array}$$
Example: Explicit Solution of a Stochastic Differential Equation

The solution of the stochastic differential equation
\[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \]
is given by
\[ X_t = X_0 \exp((\mu - \frac{1}{2} \sigma^2)t + \sigma W_t) . \]

Proof. We apply Itô’s Lemma to
\[ X_t = f(Y_t, t) := X_0 \exp((\mu - \frac{1}{2} \sigma^2)t + \sigma Y_t) \]
with \( Y_t = W_t \) and \( a \equiv 0 \), \( b \equiv 1 \).
Example: Stochastic Differential Equation for the Value of an Asset

We assume that the value of an asset is described by the random variable $S_t, t \in \mathbb{R}$, satisfying the geometric Brownian motion

$$\ln(S_t) = \ln(S_0) + (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t.$$ 

with given drift $\mu$ and given volatility $\sigma \in \mathbb{R}_+$. We note that $\ln(S_t)$ is an Itô process, since we may write

$$d(\ln(S_t)) = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t.$$ 

We apply Itô’s Lemma with $f(x) = \exp(x)$ and $a = \mu - \frac{1}{2} \sigma^2$, $b = \sigma$. Observing $\frac{\partial f}{\partial x}(\ln(S_t)) = S_t$ and $\frac{\partial^2 f}{\partial x^2}(\ln(S_t)) = S_t$ yields

$$dS_t = d(\exp(\ln(S_t))) = (\mu - \frac{1}{2} \sigma^2) S_t dt + \frac{1}{2} \sigma^2 S_t dt + \sigma S_t dW_t = \mu S_t dt + \sigma S_t dW_t.$$ 

Interpretation: The relative change $dS_t/S_t$ of the value consists of a deterministic part $\mu$ $dt$ and a stochastic part $\sigma$ $dW_t$ which represents the volatility.
3.3 Derivation of the Black-Scholes Equation for European Options

3.3.1 Basic Assumptions

We consider a financial market under the following assumptions

- The value of the asset $S_t$, $t \in \mathbb{R}_+$ satisfies the stochastic differential equation
  $$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad \mu \in \mathbb{R}, \quad \sigma \in \mathbb{R}_+.$$  

- Bonds $B_t$, $t \in \mathbb{R}_+$ are subject to the risk-free interest rate $r \in \mathbb{R}_+$, i.e., $dB_t = r B_t \, dt$.

- There are no dividends, the market is arbitrage-free, liquid and frictionless (i.e., no transaction costs, no taxes etc.).

- The asset or parts of it can be traded continuously and short-selling are allowed.

- All stochastic processes are continuous (i.e., a crash of the stock market can not be modeled).
3.3.2 Self-Financing Portfolio

We consider a portfolio consisting of $c_1(t)$ bonds $B_t$ and $c_2(t)$ assets $S_t$ as well as a European option with value $V(S_t, t)$. Assuming that the option has been sold at time $t \in \mathbb{R}_+$, the portfolio has the value

$$Y_t = c_1(t) B_t + c_2(t) S_t - V(S_t, t).$$

**Definition 3.7 (Self-Financing Portfolio)**

A portfolio

$$Y_t = c(t) \cdot S_t := \sum_{i=1}^{n} c_i(t) S_i(t).$$

consisting of $c_i(t)$ investments $S_i(t), 1 \leq i \leq n$ (bonds, stocks, options), is said to be self-financing if changes in the portfolio are only financed by either buying or selling parts of the portfolio, i.e., if for sufficiently small $\Delta t > 0$ there holds

$$Y_t = c(t) \cdot S_t = c(t - \Delta t) \cdot S_t \implies dY_t = c(t) \cdot dS_t.$$
3.3.3 Black-Scholes Equation

Theorem 3.1 (Black-Scholes Equation)
Under the previous assumptions on the financial market, let \( Y = Y_t \) be a risk-free, self-financing portfolio consisting of a bond \( B = B_t \), a stock \( S = S_t \), and a European option with value \( V = V_t \).
Then, the value \( V \) of the option satisfies the parabolic partial differential equation
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0,
\]
which is known as the Black-Scholes equation.
The final condition at time \( t = T \) (maturity date) is
\[
V(S, T) = \begin{cases} 
(S - K)^+ & \text{for a European call} \\
(K - S)^+ & \text{for a European put} 
\end{cases}
\]
The boundary conditions at \( S = 0 \) and for \( S \to \infty \) are given by
\[
V(0, t) = \begin{cases} 
0 & \text{for a European call} \\
K \exp(-r(T - t)) & \text{for a European put} 
\end{cases}
\quad V(S, t) = O(S) \quad \text{for a European call}
\quad \lim_{S \to \infty} V(S, t) = 0 \quad \text{for a European put}.
Proof. According to Itô’s Lemma, $V$ satisfies the stochastic differential equation

$$\begin{align*}
\text{(⋆)} \quad dV &= \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW.
\end{align*}$$

Inserting the stochastic differential equations for $V, B$ and $S$

$$
\begin{align*}
\quad dV &= c_1 dB + c_2 dS - dY, \quad dB = r B dt, \quad dS = \mu S dt + \sigma S dW
\end{align*}
$$

into (⋆), we obtain

$$\begin{align*}
\text{(◦)} \quad dY &= \left[ c_1 r B + c_2 \mu S - \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \right] dt + \left( c_2 \sigma S - \sigma S \frac{\partial V}{\partial S} \right) dW.
\end{align*}$$
The assumption of a risk-free portfolio implies the non-existence of stochastic fluctuations, i.e.,
the coefficient in front of $dW$ has to be set to zero: $c_2 = \frac{\partial V}{\partial S}$.

On the other hand, the assumption of a risk-free portfolio implies

$$dY = r\ Y\ dt = r\ (c_1\ B + c_2\ S - V)\ dt.$$  

Now, substituting (oo) into (o) yields

$$r\ (c_1\ B + \frac{\partial V}{\partial S}\ S - V)\ dt = (c_1\ r\ B + c_2\ \mu\ S - \frac{\partial V}{\partial t} - \mu\ S \frac{\partial V}{\partial S} - \frac{1}{2}\ \sigma^2\ S^2 \frac{\partial^2 V}{\partial S^2})\ dt =$$

$$= (c_1\ r\ B - \frac{\partial V}{\partial t} - \frac{1}{2}\ \sigma^2\ S^2 \frac{\partial^2 V}{\partial S^2})\ dt.$$  

The coefficients in front of $dt$ must be equal, and hence, we obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2\ S^2 \frac{\partial^2 V}{\partial S^2} + r\ S \frac{\partial V}{\partial S} - r\ V = 0.$$
We note that the **final condition** at \( t = T \) follows from Chapter 1.

In order to derive the **boundary conditions** at \( S = 0 \) and for \( S \to \infty \), we also have to distinguish between a call and a put:

**Case 1: European call**

For \( S = 0 \), i.e., a stock with zero value, the option to buy such a stock is of value zero as well. On the other hand, for \( S \gg 1 \), i.e., for a high value of the stock, it is almost sure that the option will be exercised, and we have \( V \approx S - K \exp(-r(T-t)) \). Since the strike \( K \) can be neglected for large \( S \), we thus obtain \( V = O(S) \ (S \to \infty) \).

**Case 2: European put**

For \( S \gg 1 \) it is unlikely that the put \( V = P \) will be exercised, i.e., \( P(S,t) \to 0 \) as \( S \to \infty \).

On the other hand, at \( S = 0 \), the **put-call parity** (Theorem 1.1) implies

\[
P(0,t) = (C(S,t) + K \exp(-r(T-t)) - S)|_{S=0} = K \exp(-r(T-t)).
\]
Theorem 3.2 (Black-Scholes Formula for European Calls)
For a European call, the Black-Scholes equation with boundary data and final condition as in Theorem 3.1 has the explicit solution

\[ \text{(\star)} \quad V(S,t) = S \Phi(d_1) - K \exp(-r(T-t)) \Phi(d_2), \]

where \( \Phi \) denotes the distribution function of the standard normal distribution

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-s^2/2) \, ds , \quad x \in \mathbb{R} \]

and \( d_{\nu}, 1 \leq \nu \leq 2 \), are given by

\[ d_{1/2} = \frac{\ln(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}. \]
Proof. The idea of proof is to transform the Black-Scholes equation to $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ for which an analytical solution is available and then to transform back to the original variables.

Step 1: Transformation

We perform the transformation of variables

$$x = \ln(S/K) \ , \ \tau = \frac{1}{2} \sigma^2 (T - t) \ , \ v(x, \tau) = \frac{V(S, t)}{K}.$$ 

Obviously, $x \in \mathbb{R}$ (since $S > 0$), $0 \leq \tau \leq T_0 := \sigma^2 T / 2$ (since $0 \leq t \leq T$), and $v(x, \tau) \geq 0$ (since $V(S, t) \geq 0$). Moreover, the chain rule implies

$$\frac{\partial V}{\partial t} = K \frac{\partial v}{\partial t} = K \frac{\partial v}{\partial \tau} \frac{d\tau}{dt} = -\frac{1}{2} \sigma^2 K \frac{\partial v}{\partial \tau},$$

$$\frac{\partial V}{\partial S} = K \frac{\partial v}{\partial x} \frac{dx}{dS} = \frac{K}{S} \frac{\partial v}{\partial x},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S} \left( \frac{K}{S} \frac{\partial v}{\partial x} \right) = -\frac{K}{S^2} \frac{\partial v}{\partial x} + \frac{K}{S} \frac{\partial^2 v}{\partial x^2} \frac{1}{S} = \frac{K}{S^2} (-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}).$$
Hence, the transformed Black-Scholes equation takes the form

\[-\frac{\sigma^2}{2} K \frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2} S^2 \frac{K}{S^2} \left( - \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) + r S \frac{K}{S} \frac{\partial v}{\partial x} - r K v = 0 .\]

Setting \( \kappa := \frac{2r}{\sigma^2} \) and \( T_0 := \frac{\sigma^2 T}{2} \), it follows that

\[(+) \quad \frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + (1 - \kappa) \frac{\partial v}{\partial x} + \kappa v = 0 , \quad x \in \mathbb{R} , \quad \tau \in (0, T_0] \]

with initial condition (observe \( (S - K)^+ = K(\exp(x) - 1)^+ \))

\[v(x, 0) = (\exp(x) - 1)^+ , \quad x \in \mathbb{R} .\]

For the elimination of \( \partial v/\partial x \) and \( v \), we use the ansatz

\[v(x, \tau) = \exp(\alpha x + \beta \tau) \ u(x, \tau) , \quad \alpha, \beta \in \mathbb{R} .\]

Inserting this ansatz into \((+)\) and dividing by \( \exp(\alpha x + \beta \tau) \) yields

\[\beta u + \frac{\partial u}{\partial \tau} - \alpha^2 u - 2 \alpha \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + (1 - \kappa)(\alpha u + \frac{\partial u}{\partial x}) + \kappa u = 0 .\]
Choosing \( \alpha, \beta \in \mathbb{R} \) such that
\[
\beta - \alpha^2 + (1 - \kappa) \alpha + \kappa = 0 \\
-2\alpha + (1 - \kappa) = 0
\]
\[
\implies \alpha = -\frac{1}{2} (\kappa - 1) \\
\beta = -\frac{1}{4} (\kappa + 1)^2
\]

we find that the function
\[
u(x, \tau) = \exp\left(\frac{1}{2} (\kappa - 1) x + \frac{1}{4} (\kappa + 1)^2 \tau\right) v(x, \tau)
\]
satisfies the linear diffusion equation
\[
\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0 , \quad x \in \mathbb{R} , \ \tau \in (0, T_0]
\]
with the initial condition
\[
u(x, 0) = \nu_0(x) := \exp((\kappa - 1) x/2) \ (\exp(x) - 1)^+ = (\exp((\kappa + 1)x/2) - \exp((\kappa - 1)x/2))^+ , \quad x \in \mathbb{R} .
\]

**Step 2:** Analytical solution of the linear diffusion equation

The analytical solution of the linear diffusion equation is given by
\[
u(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{+\infty} \nu_0(s) \exp(-\frac{(x - s)^2}{4\tau}) \, ds .
\]
For the evaluation of the integral we use the transformation of variables \( y = (s - x)/\sqrt{2\tau} \).

Together with the expression for the initial condition \( u_0 = u(x, 0) \) we obtain

\[
    u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0 \left( \sqrt{2\tau} \ y + x \right) \exp(-y^2/2) \, dy = \\
    = \frac{1}{\sqrt{2\pi}} \left[ \int_{-x/\sqrt{2\tau}}^{+\infty} \exp(\frac{1}{2}(\kappa + 1)(x + y\sqrt{2\tau}))\exp(-y^2/2) \, dy - \int_{-x/\sqrt{2\tau}}^{-\infty} \exp(\frac{1}{2}(\kappa - 1)(x + y\sqrt{2\tau}))\exp(-y^2/2) \, dy \right].
\]

Using the representation

\[
    \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{+\infty} \exp(\frac{1}{2}(\kappa \pm 1)(x + y\sqrt{2\tau}))\exp(-y^2/2) \, dy = \exp(\frac{1}{2}(\kappa \pm 1)x + \frac{1}{4}(\kappa \pm 1)^2 \tau) \Phi(d_{1/2}),
\]

it follows that

\[
    u(x, \tau) = \exp(\frac{1}{2}(\kappa + 1)x + \frac{1}{4}(\kappa + 1)^2 \tau) \Phi(d_1) - \exp(\frac{1}{2}(\kappa - 1)x + \frac{1}{4}(\kappa - 1)^2 \tau) \Phi(d_2).
\]
Step 3: Back-Transformation

In view of \( v(x, \tau) = V(S, t)/K \) and \( v(x, \tau) = \exp(-\frac{1}{2}(\kappa - 1)x - \frac{1}{4}(\kappa + 1)^2\tau)u(x, \tau) \) we obtain

\[
V(S, t) = K v(x, \tau) = K \exp(-\frac{1}{2}(\kappa - 1)x - \frac{1}{4}(\kappa + 1)^2\tau) u(x, \tau) .
\]

Now, inserting

\[
u(x, \tau) = \exp(\frac{1}{2}(\kappa + 1)x + \frac{1}{4}(\kappa + 1)^2\tau) \Phi(d_1) - \exp(\frac{1}{2}(\kappa - 1)x + \frac{1}{4}(\kappa - 1)^2\tau) \Phi(d_2) ,
\]

it follows that

\[
V(S, t) = K \exp(x) \Phi(d_1) - K \exp(-\frac{1}{4}(\kappa + 1)^2\tau + \frac{1}{4}(\kappa - 1)^2\tau) \Phi(d_2) .
\]

Finally, observing \( x = \ln(S/K) \) and \( \tau = \sigma^2(T-t)/2 \), we arrive at

\[
V(S, t) = S \Phi(d_1) - K \exp(-r(T-t)) \Phi(d_2) .
\]
Step 4: Checking the Boundary Conditions and the Final Condition

(i) Boundary condition at $S = 0$
In view of $d_{1/2} \to -\infty$ for $S \to 0$, we have $\Phi(d_{1/2}) \to 0$ for $S \to 0$ whence

$$V(S, t) = S \Phi(d_1) - K \exp(-r(T-t)) \Phi(d_2) \to 0 \quad \text{as} \quad S \to 0.$$  

(ii) Boundary condition for $S \to +\infty$
Observing $\Phi(d_1) \to 1$ and $\Phi(d_2)/S \to 0$ for $S \to +\infty$, we get

$$\frac{V(S, t)}{S} = \frac{\Phi(d_1) - K \exp(-r(T-t)) \Phi(d_2)}{S} \to 1 \quad \text{as} \quad S \to +\infty.$$  

(iii) Final condition at $t = T$
For $t \to T$ we get

$$\frac{\ln(S/K)}{\sigma \sqrt{T-t}} \to \begin{cases} +\infty & , \quad S > K \\ 0 & , \quad S = K \\ -\infty & , \quad S < K \end{cases} \quad \Rightarrow \quad \Phi(d_{1/2}) \to \begin{cases} 1 & , \quad S > K \\ 1/2 & , \quad S = K \\ 0 & , \quad S < K \end{cases}$$  

$$V(S, T) \to \begin{cases} S - K & , \quad S > K \\ 0 & , \quad S \leq K \end{cases} = (S - K)^+. $$
Theorem 3.3 (Black-Scholes Formula for European Puts)

For a European put, the Black-Scholes equation with boundary data and final condition as in Theorem 3.1 has the explicit solution

\[(**) \quad V(S,t) = K \exp(-r(T-t)) \Phi(-d_2) - S \Phi(-d_1), \]

where \( \Phi \) and \( d_{1/2} \) are given as in Theorem 3.2.

**Proof.** The proof follows readily from Theorem 3.2 and the put-call parity (cf. Theorem 1.1).
Black-Scholes Formula for a European Call and a European Put

Values of a European call (left) and a European put (right) in dependence of $S$ at times $0 \leq t \leq 1$ for $K = 100$, $T = 1$, $r = 0.1$, $\sigma = 0.4$ (courtesy of [1])
3.3.4 Interpretation of the Option Price as a Discounted Expectation

Consider a European call option \( V = V(S,t) \) with \( \Lambda(S) = (S - K)^+ \) and the proof of the Black-Scholes formula (Theorem 3.2): In Step 3, the back transformation can be performed based on the representation

\[
u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(\sqrt{2\tau} y + x) \exp(-y^2/2) \, dy .
\]

Using further \( \tilde{S} := \exp(\sqrt{2\tau} y) \), we find

\[
V(S,t) = \frac{K}{\sqrt{2\pi}} \exp\left(-\frac{(\kappa - 1)x/2 - (\kappa + 1)^2\tau/4}{\sqrt{2\tau}}\right) \int_{\mathbb{R}} u_0(\sqrt{2\tau} y + x) \exp(-y^2/2) \, dy = \\
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{(\kappa + 1)^2\tau/4 + (\kappa - 1)\sqrt{2\tau}y/2 - y^2/2}{2\sigma^2(T - t)}\right) (\exp(\sqrt{2\tau} y) S - K)^+ \, dy = \\
= \exp(-r(T - t)) \, E(\Lambda(S)) , \quad E(\Lambda(S)) = \int_{0}^{\infty} \tilde{S} f(\tilde{S}; S, t) \, \Lambda(\tilde{S}) \, d\tilde{S} ,
\]

where \( E(\Lambda(S)) \) is the expectation of \( \Lambda(S) \) w.r.t. the density function of the log-normal distribution

\[
f(\tilde{S}; S, t) = \frac{1}{\tilde{S}\sigma\sqrt{2\pi(T - t)}} \exp\left(-\frac{(\ln(\tilde{S}/S) - (r - \sigma^2/2)(T - t))^2}{2\sigma^2(T - t)}\right) .
\]
3.4 Numerical Evaluation of the Black-Scholes Formula

The evaluation of the Black-Scholes formula requires a numerically efficient approximation of the error function \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt \). This can be done by either rational best approximation in the nonlinear least-squares sense or piecewise polynomial approximation (see Chapters 3.5 and 4.2, Handout Numerical Analysis I, Fall 2005, on my webpage).

3.4.1 Rational Best Approximation of the error function \( \text{erf} \)

Recalling the asymptotic properties

\[
\lim_{x \to \infty} \text{erf}(x) = 1, \quad \lim_{x \to \infty} \frac{1 - \text{erf}(x)}{\text{erf}(x)} = \lim_{x \to \infty} \int_x^\infty \exp(x^2 - t^2) \, dt = 0,
\]

we look for a rational function \( \text{erf}^R \)

\[
\frac{1 - \text{erf}^R}{\text{erf}(x)} = a_1 \eta + a_2 \eta^2 + a_3 \eta^3, \quad \eta := \frac{1}{1 + px}
\]

such that \( \text{erf}^R(0) = 0 \) resulting in, e.g., \( a_3 = \sqrt{\pi}/2 - a_1 - a_2 \). We determine \( \text{erf}^* \) as the best \( L^2 \)-approximation of \( \text{erf} \) within the class of functions \( \text{erf}^R \)

\[
(\diamond) \quad \| \text{erf}^* - \text{erf} \|_{L^2(0, \infty)}^2 = \inf_{a_1, a_2, p} \| \text{erf}^R - \text{erf} \|_{L^2(0, \infty)}^2 = \inf_{a_1, a_2, p} \int_0^\infty |\text{erf}^R(x) - \text{erf}(x)|^2 \, dx.
\]
Obviously, (⋄) represents an infinite dimensional nonlinear least-squares problem. We reduce it to a computationally more accessible finite dimensional nonlinear least-squares problem by approximating the integral with respect to a partition \( \Delta = \{0 = x_0 < x_1 < \ldots < x_n < \infty\} \) of the domain of integration

\[
(+) \quad \min_{z=(a_1,a_2,p)} n^{-1} \| F(z) \|_2^2, \quad F(z) := \left( \begin{array}{c} \text{erf}^R(x_1) - \text{erf}(x_1) \\ \cdot \\ \text{erf}^R(x_n) - \text{erf}(x_n) \end{array} \right),
\]

where \( \| \cdot \|_2 \) stands for the Euclidean norm in \( \mathbb{R}^n \).

Given a start iterate \( z^{(0)} \in \mathbb{R}^3 \), the Gauss-Newton method for the solution of (\( + \)) is given by

\[
F'(z^{(k)})^T F'(z^{(k)}) \Delta z^{(k)} = - F'(z^{(k)})^T F(z^{(k)}),
\]

\[
z^{(k+1)} = z^{(k)} + \Delta z^{(k)}.
\]
3.4.2 Piecewise Polynomial Interpolation

Another way to evaluate the error function is to approximate \( \text{erf} \) on \( [0, x_{\text{max}}] \) by a piecewise polynomial interpolation using either cubic Hermite interpolation or the complete cubic spline interpoland with respect to a partition \( \Delta = \{0 = x_0 < x_1 < \ldots < x_n = x_{\text{max}}\} \).

The cubic Hermite interpoland \( h_\Delta \in C^1([0, x_{\text{max}}]) \) with \( h_\Delta|_{[x_i, x_{i+1}]} \in P_3([x_i, x_{i+1}]), 0 \leq i \leq n - 1 \), is given by

\[
h_\Delta(x_j) = \text{erf}(x_j), \quad h_\Delta'(x_j) = \text{erf}_x(x_j), \quad j \in \{i, i+1\}
\]

and has the representation

\[
h_\Delta(x) = \text{erf}(x_i) \varphi_1(t) + \text{erf}(x_{i+1}) \varphi_2(t) + h_i \text{erf}_x(x_i) \varphi_3(t) + h_i \text{erf}_x(x_{i+1}) \varphi_4(t), \quad t := \frac{x - x_i}{h_i}, \quad h_i := x_{i+1} - x_i,
\]

\[
\varphi_1(t) := 1 - 3t^2 + 2t^3, \quad \varphi_2(t) := 3t^2 - 2t^3, \quad \varphi_3(t) := t - 2t^2 + t^3, \quad \varphi_4(t) := -t^2 + t^3.
\]

Having determined \( h_\Delta \), we define \( \text{erf}^* \) by \( \text{erf}^*(x) = h_\Delta(x), x \in [0, x_{\text{max}}] \), and \( \text{erf}^*(x) = h_\Delta(x_{\text{max}}), x > x_{\text{max}} \).

For the complete cubic spline interpoland we refer to Chapt. 3.5, Handout Numer. Anal. II.
Approximation of the error function $\text{erf}$ by cubic Hermite interpolation ($x_{\text{max}} = 3$, $n = 8$).
3.5 Greeks and Volatility

3.5.1 Greeks
Let $V$ be the value of a call option or a put option. Greeks are derivatives of $V$ and hence can be used for a hedging of the portfolio.

Definition 3.8 (Greeks)
The Greeks $\Delta$, $\Gamma$, Vega (resp. Kappa) $\kappa$, Theta $\Theta$, and Rho $\rho$ are defined by

\[
\Delta := \frac{\partial V}{\partial S} , \quad \Gamma := \frac{\partial^2 V}{\partial S^2} , \quad \kappa := \frac{\partial V}{\partial \sigma} , \quad \Theta := \frac{\partial V}{\partial t} , \quad \rho := \frac{\partial V}{\partial r} .
\]

Theorem 3.4 (Computation of Greeks)
The Greeks have the following representations

\[
\Delta = \Phi(d_1) , \quad \Gamma = \frac{\Phi'(d_1)}{S} \sigma \sqrt{T-t} , \quad \kappa = S \sqrt{T-t} \Phi'(d_1) ,
\]
\[
\Theta = - S \sigma \frac{\Phi'(d_1)}{2} \sqrt{T-t} - r K \exp(-r(T-t)) \Phi(d_2) , \quad \rho = (T-t) K \exp(-r(T-t)) \Phi(d_2) .
\]
Greeks at $t = 0$ (dotted line), $t = 0.4$ (straight line) and $t = 0.8$ (bold line) for $K = 100$, $T = 1$, $r = 0.1$, $\sigma = 0.4$ (courtesy of [1])
3.5.2 Volatility

The Black-Scholes formulas reveal that the price of options depends on the volatility \( \sigma \) of the basic asset, but not on the drift \( \mu \). Since \( \sigma \) is only known in the past, we need an efficient prediction of \( \sigma \) for the future. The most commonly used predictors are the historical volatility and the implicit volatility.

3.5.2.1 Historical Volatility

The historical volatility is the annualized standard deviation of the logarithmic changes in the value of the asset.

**Definition 3.8 (Historical Volatility)**

Let \( S_i, 1 \leq i \leq n \), be the value of the asset on the day \( t_i \) and denote by \( N \) the average number of trading days at the stock market. Then, the historical volatility \( \sigma_{\text{hist}} \) is defined as

\[
\sigma_{\text{hist}} = \sqrt{N} \left( \frac{1}{n-1} \sum_{i=1}^{n-1} [(\ln S_{i+1} - \ln S_i) - \frac{1}{n-1} \sum_{i=1}^{n-1} (\ln S_{i+1} - \ln S_i)]^2 \right)^{1/2}.
\]
3.5.2.2 Implicit Volatility

Consider a European call option $C = C(t)$ and assume that the prize at some time $t_0 < T$ is known and given by $C_0 = C(t_0)$. The Black-Scholes formula for European call options (Theorem 3.2) shows that $C$ depends on the volatility $\sigma$ according to

$$C(\sigma) = S \Phi(d_1(\sigma)) - K \exp(-r(T-t)) \Phi(d_2(\sigma)).$$

**Definition 3.9 (Implicit Volatility)**

Assume that $C_0$ satisfies the arbitrage estimate $(S - K \exp(-r(T-t))^+ \leq C_0 \leq S$. Then, the equation $C(\sigma_{imp}) = C_0$ admits a unique solution $\sigma_{imp} > 0$ which is called the implicit volatility.

$$C(\sigma) = S \Phi(d_1(\sigma)) - K \exp(-r(T-t)) \Phi(d_2(\sigma)).$$

Given a start iterate $\sigma^{(0)} > 0$, the implicit volatility can be computed by Newton’s method

$$\sigma^{(k+1)} = \sigma^{(k)} - \frac{C^{(k)} - C_0}{C' \sigma^{(k)}} = \sigma^{(k)} - \frac{C^{(k)} - C_0}{\kappa(\sigma^{(k)})} \quad \text{,} \quad \kappa(\sigma^{(k)}) := S \sqrt{T-t} \Phi'(d_1(\sigma^{(k)})).$$