The Determinant of a Product and Transpose

Recall the Leibniz formula for the determinant. Given an $m \times m$ matrix A, we have

$$\det(A) \equiv \sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, a_{1,j_1} a_{2,j_2} \cdots a_{m,j_m},$$

where $\mathbf{j} = j_1, j_2, \ldots, j_m$ represents one of the $m!$ permutations of $1, 2, \ldots, m$, P_m is the set of all such permutations, and $\operatorname{sgn}(\mathbf{j})$ equals $+1$ or -1 depending on whether the given permutation $\mathbf{j}$ is even or odd.

Here, I'll first derive the well-known formula for the determinant of a matrix product. Given $m \times m$ matrices A and B, we'll show

$$\det(AB) = \det(A)\det(B).$$

To see this, recall how matrix multiplication is defined

$$(AB)_{i,j} = \sum_{k=1}^{m} a_{i,k}\, b_{k,j}.$$

Therefore

$$\begin{aligned}
\det(AB) &= \sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, (AB)_{1,j_1} (AB)_{2,j_2} \cdots (AB)_{m,j_m} \\
&= \sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j}) \left(\sum_{k_1=1}^{m} a_{1,k_1} b_{k_1,j_1} \right) \left(\sum_{k_2=1}^{m} a_{2,k_2} b_{k_2,j_2} \right) \cdots \left(\sum_{k_m=1}^{m} a_{m,k_m} b_{k_m,j_m} \right).
\end{aligned}$$

Distribute this out to get

$$\begin{aligned}
\det(AB) &= \left(\sum_{k_1=1}^{m} \cdots \sum_{k_m=1}^{m} \right) \left(\sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, a_{1,k_1} b_{k_1,j_1} a_{2,k_2} b_{k_2,j_2} \cdots a_{m,k_m} b_{k_m,j_m} \right) \\
&= \left(\sum_{k_1=1}^{m} \cdots \sum_{k_m=1}^{m} \right) \left((a_{1,k_1} a_{2,k_2} \cdots a_{m,k_m}) \sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, b_{k_1,j_1} b_{k_2,j_2} \cdots b_{k_m,j_m} \right).
\end{aligned}$$

Now, here's what makes things work. Notice that for any sequence $k_1, k_2 \ldots, k_m \notin \mathrm{P}_m$ we must have

$$\sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, b_{k_1,j_1} b_{k_2,j_2} \cdots b_{k_m,j_m} = 0.$$

This is true because in this case the sum above represents the determinant of a matrix that has at least two identical rows. Furthermore, when $k_1, k_2 \ldots, k_m \equiv \mathbf{k} \in \mathrm{P}_m$ we have

$$\sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, b_{k_1,j_1} b_{k_2,j_2} \cdots b_{k_m,j_m} = \operatorname{sgn}(\mathbf{k}) \sum_{\mathbf{j} \in \mathrm{P}_m} \operatorname{sgn}(\mathbf{j})\, b_{1,j_1} b_{2,j_2} \cdots b_{m,j_m}.$$

This is true because here the sum above represents the determinant of a matrix that's formed from B by interchanging rows either an odd number to times, $\text{sgn}(\mathbf{k}) = -1$, or an even number of times, $\text{sgn}(\mathbf{k}) = +1$. Taking these two facts into account, we've shown

$$\begin{aligned}\det(AB) &= \sum_{\mathbf{k}\in \mathrm{P}_m} \left((a_{1,k_1} a_{2,k_2} \cdots a_{m,k_m})\ \text{sgn}(\mathbf{k}) \sum_{\mathbf{j}\in \mathrm{P}_m} \text{sgn}(\mathbf{j})\, b_{1,j_1} b_{2,j_2} \cdots b_{m,j_m} \right) \\ &= \sum_{\mathbf{k}\in \mathrm{P}_m} \text{sgn}(\mathbf{k})\, a_{1,k_1} a_{2,k_2} \cdots a_{m,k_m} \sum_{\mathbf{j}\in \mathrm{P}_m} \text{sgn}(\mathbf{j})\, b_{1,j_1} b_{2,j_2} \cdots b_{m,j_m} \\ &= \det(A)\det(B).\end{aligned}$$

Now I'll derive the fact that

$$\det(A^T) = \det(A).$$

Recall how the transpose is defined

$$(A^T)_{i,j} = a_{j,i}.$$

Plug this into the Leibniz determinant formula to get

$$\det(A^T) = \sum_{\mathbf{j}\in \mathrm{P}_m} \text{sgn}(\mathbf{j})\, (A^T)_{1,j_1} (A^T)_{2,j_2} \cdots (A^T)_{m,j_m} = \sum_{\mathbf{j}\in \mathrm{P}_m} \text{sgn}(\mathbf{j})\, a_{j_1,1} a_{j_2,2} \cdots a_{j_m,m}.$$

For each product term within the sum on the right, rearrange the order of multiplication so that

$$a_{j_1,1} a_{j_2,2} \cdots a_{j_m,m} = a_{1,j'_1} a_{2,j'_2} \cdots a_{m,j'_m}.$$

For example ($m = 3$)

$$\mathbf{j} = 3,1,2 \quad \Rightarrow \quad a_{3,1} a_{1,2} a_{2,3} = a_{1,2} a_{2,3} a_{3,1} \quad \Rightarrow \quad \mathbf{j}' = 2,3,1.$$

In other words, find $\mathbf{j}' = j'_1, j'_2, \ldots, j'_m \in \mathrm{P}_m$ such that for each $i = 1, 2, \ldots, m$ we have $j_{j'_i} = i$. One can show that $\mathbf{j}'$ can be obtained from $\mathbf{j}$ by an even number of interchanges. Therefore, $\text{sgn}(\mathbf{j}) = \text{sgn}(\mathbf{j}')$, and so we conclude

$$\det(A^T) = \sum_{\mathbf{j}\in \mathrm{P}_m} \text{sgn}(\mathbf{j})\, a_{j_1,1} a_{j_2,2} \cdots a_{j_m,m} = \sum_{\mathbf{j}'\in \mathrm{P}_m} \text{sgn}(\mathbf{j}')\, a_{1,j'_1} a_{2,j'_2} \cdots a_{m,j'_m} = \det(A).$$