The Cofactor Expansion, Cramer's Rule and Eigenvalues

The *cofactor expansion*, sometimes called the *Laplace expansion* named after Pierre-Simon Laplace, is one way to view and compute the determinant. Here's how it works. Let A be an $m \times m$ matrix and from it define its i, jth *minor matrix*, say $M_{i,j}$, as the $(m-1) \times (m-1)$ matrix formed by removing A's ith row and jth column. For example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad M_{1,2} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}, \quad M_{2,3} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}.$$

Now, for any <u>row</u>, say i_*, the cofactor expansion says

$$\det(A) = \sum_{j=1}^{m} a_{i_*,j} \, (-1)^{i_*+j} \det(M_{i_*,j}).$$

FYI: The number $c_{i,j} \equiv (-1)^{i+j} \det(M_{i,j})$ is called the i, jth *cofactor* of A.

Using the first row in the example above, we find

$$\begin{aligned} &\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\ &= 1 \, (-1)^{1+1} \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} + 2 \, (-1)^{1+2} \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \, (-1)^{1+3} \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= +1 \, (45 - 48) - 2 \, (36 - 42) + 3 \, (32 - 35) = 0. \end{aligned}$$

Since $\det(A) = \det(A^T)$, we can also use a cofactor expansion along any <u>column</u> j_*

$$\det(A) = \sum_{i=1}^{m} a_{i,j_*} \, (-1)^{i+j_*} \det(M_{i,j_*}).$$

Using the second column in the example above, we find

$$\begin{aligned} &\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\ &= 2 \, (-1)^{1+2} \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 5 \, (-1)^{2+2} \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} + 8 \, (-1)^{3+2} \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} \\ &= -2 \, (36 - 42) + 5 \, (9 - 21) - 8 \, (6 - 12) = 0. \end{aligned}$$

When compared to Leibniz's formula for the determinant, these cofactor expansions offer an easy to remember formula you can use to compute $\det(A)$ when A is 4×4 or larger. To reduce your work, pick the row or column that has the greatest number of zeros in it.

Here's a 4×4 example. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{pmatrix},$$

and cofactor along the third column

$$\begin{aligned} \det(A) &= 0\,(-1)^{1+3}\det\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} + 1\,(-1)^{2+3}\det\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \\ &\quad + 2\,(-1)^{3+3}\det\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} + 0\,(-1)^{4+3}\det\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}. \end{aligned}$$

There's no need to compute the first and fourth 3×3 determinants on the right since they are both multiplied by zero. So

$$\begin{aligned} \det(A) &= 1\,(-1)^{2+3}\det\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} + 2\,(-1)^{3+3}\det\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \\ &= -1\,(1+6+2-4-3-1) + 2\,(1+4-4-2) = -3. \end{aligned}$$

I'm now going to derive Laplace's cofactor formulae, but I'll need to first establish a preliminary basic fact. Suppose an $n \times n$ matrix A takes the particular "block" form

$$A = \begin{pmatrix} (\,M\,) & (\,a\,) \\ (\,0\,) & 1 \end{pmatrix},$$

where $(\,M\,)$ denotes an $(n-1) \times (n-1)$ block, $(\,a\,)$ denotes a $(n-1) \times 1$ column block, $(\,0\,)$ denotes a $1 \times (n-1)$ row block which contains all zeros and 1 denotes the scalar one. Then

$$\det(A) = \det(M).$$

To see this is true, recall Leibniz's definition of the determinant

$$\det(A) = \sum_{\mathbf{j} \in \mathcal{P}_n} \operatorname{sgn}(\mathbf{j})\,(A_{1,j_1} \cdots A_{n,j_n}),$$

and observe for the rightmost term in the product above

$$A_{n,j_n} = \begin{cases} 0 & \text{if } j_n < n \\ 1 & \text{if } j_n = n. \end{cases}$$

So it's easy to see

$$\det(A) = \sum_{\substack{\mathbf{j} \in \{\mathcal{P}_n \\ : j_n = n\}}} \operatorname{sgn}(\mathbf{j})\,\left(A_{1,j_1} \cdots A_{n-1,j_{n-1}}\right).$$

However, for $\mathbf{j} \in \{\mathcal{P}_n : j_n = n\}$, check that $A_{i,j_i} = M_{i,j_i}$ for every $1 \le i \le n-1$ and also $\operatorname{sgn}(\mathbf{j}) = \operatorname{sgn}(j_1, \ldots, j_{n-1}, n) = \operatorname{sgn}(j_1, \ldots, j_{n-1}) \equiv \operatorname{sgn}(\mathbf{j}')$ where $\mathbf{j}' \in \mathcal{P}_{n-1}$. Therefore

$$\det(A) = \sum_{\mathbf{j}' \in \mathcal{P}_{n-1}} \operatorname{sgn}(\mathbf{j}') \left(M_{1,j'_1} \cdots M_{n-1,j'_{n-1}} \right) = \det(M).$$

Next, I'll use this basic fact together with certain properties of the determinant from your previous homework to derive Laplace's "row" expansions. To this end, let A now denote a general $n \times n$ matrix and i_* any row index between 1 and n. Write A by rows, i.e.

$$A = \begin{pmatrix} (\, a_1 \,) \\ \vdots \\ (\, a_{i_*} \,) \\ \vdots \\ (\, a_n \,) \end{pmatrix},$$

where $(\, a_i \,) \equiv (\, a_{i,1} \quad \cdots \quad a_{i,n} \,)$ denotes the ith row of A. As done earlier for $\mathbb{R}^n$'s standard basis, observe here we can write

$$(\, a_{i_*} \,) = \sum_{k=1}^{n} a_{i_*,k} \, (\, e_k \,),$$

where $(\, e_k \,) \equiv (\, 0 \cdots 0 \; 1 \; 0 \cdots 0 \,)$; the 1 is in the kth column of $(\, e_k \,)$. With this notation,

$$\det(A) = \det \begin{pmatrix} (\, a_1 \,) \\ \vdots \\ (\, a_{i_*} \,) \\ \vdots \\ (\, a_n \,) \end{pmatrix} = \det \begin{pmatrix} (\, a_1 \,) \\ \vdots \\ \sum_{k=1}^{m} a_{i_*,k} \, (\, e_k \,) \\ \vdots \\ (\, a_n \,) \end{pmatrix} = \sum_{k=1}^{n} a_{i_*,k} \det \begin{pmatrix} (\, a_1 \,) \\ \vdots \\ (\, e_k \,) \\ \vdots \\ (\, a_n \,) \end{pmatrix},$$

where the identity on the right follows because the determinant is multilinear by rows, see the 5th bulleted item on your previous homework. Also recall exchanging rows or columns changes the sign of the determinant, see the 3rd bulleted item, so you can use $n - i_*$ row interchanges and $n - k$ column interchanges to conclude

$$\det \begin{pmatrix} (\, a_1 \,) \\ \vdots \\ (\, e_k \,) \\ \vdots \\ (\, a_n \,) \end{pmatrix} = (-1)^{n-i_*} (-1)^{n-k} \det \begin{pmatrix} (\, M_{i_*,k} \,) & (\, a \,) \\ (\, 0 \,) & 1 \end{pmatrix},$$

where $(\, M_{i_*,k} \,)$ is the i_*, kth minor matrix of A, and the column block $(\, a \,)$ contains elements of A's kth column. Finally, since $(-1)^{n-i_*}(-1)^{n-k} = (-1)^{i_*+k}$, this together with the sum displayed on the right side directly above and the preliminary basic fact combine to yield the sought for cofactor expansion formula for arbitrary row index i_*.

The Swiss mathematician Gabriel Cramer was the first to publish the relation between the solution of a square linear system and associated determinants. His observation has come to be known as *Cramer's rule.*

First let me state Cramer's rule for finding an inverse matrix. Let C denote the *cofactor matrix* for an $m \times m$ matrix A. Each element of C is given by

$$c_{i,j} \equiv (-1)^{i+j} \det(M_{i,j}) \text{ for each } 1 \le i \le m,\ 1 \le j \le m.$$

Then, if $\det(A) \neq 0$, Cramer's rule gives

$$A^{-1} = \frac{1}{\det(A)}\, C^T.$$

This formula should not in general be regarded as a practical tool for computing inverses since working out cofactors can be very time consuming. It is handy when A is small however. For example, consider the 2×2 matrix A (assume $\det(A) \neq 0$)

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \Rightarrow \quad C = \begin{pmatrix} a_{2,2} & -a_{2,1} \\ -a_{1,2} & a_{1,1} \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}.$$

Here, C^T is found by flipping the diagonal entries of A and then changing the signs of the off–diagonal terms. Many students will memorize this 2×2 formula.

The derivation of Cramer's inverse formula is really pretty easy. Recall how matrix multiplication and cofactors are defined

$$(A\,C^T)_{i,j} = \sum_{k=1}^{m} a_{i,k}\, c^T_{k,j} = \sum_{k=1}^{m} a_{i,k}\, c_{j,k} = \sum_{k=1}^{m} a_{i,k}\, (-1)^{j+k} \det(M_{j,k}).$$

Now return to the 'row' cofactor expansion formula for the determinant and notice

$$\sum_{k=1}^{m} a_{i,k}\, (-1)^{j+k} \det(M_{j,k}) = \det(A'),$$

where the matrix A' is obtained from A by replacing its jth row by its ith row. Therefore, since a matrix that has two identical rows has determinant zero, see that when $i \neq j$ we have $\det(A') = 0$. When $i = j$ clearly $\det(A') = \det(A)$. In other words

$$(A\,C^T)_{i,j} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad A\,C^T = \det(A)\, I,$$

and from this we easily see that Cramer's inverse formula is valid.

An alternate formulation of Cramer's rule gives the solution of $A\mathbf{x} = \mathbf{b}$ in terms of determinants.

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\,\mathbf{b} = \frac{1}{\det(A)}\, C^T\,\mathbf{b} \quad \Rightarrow \quad x_i = \frac{1}{\det(A)} \sum_{k=1}^{m} c_{k,i}\, b_k.$$

Notice above that $\sum_{k=1}^{m} c_{k,i}\, b_k = \sum_{k=1}^{m} b_k\, (-1)^{k+i} \det(M_{k,i})$ is the cofactor expansion for the determinant of the matrix formed by replacing A's ith column with the column vector $\mathbf{b}$. That is

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad x_i = \frac{\det(B_i)}{\det(A)},$$

where the matrix B_i is the same as A except its ith column is replaced by vector $\mathbf{b}$. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad x_1 = \frac{1}{4-6} \det \begin{pmatrix} 5 & 2 \\ 6 & 4 \end{pmatrix} = \frac{8}{-2} = -4, \quad x_2 = \frac{1}{4-6} \det \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix} = \frac{-9}{-2} = 9/2.$$

You might also want to consider memorizing this form of Cramer's rule long term.

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1. Use a cofactor expansion to evaluate the determinant of the following.

(a) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

2. Use a cofactor expansion to evaluate the determinant of the following.

(a) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 2 & 0 & 1 & 3 \end{pmatrix}$

3. Use Cramer's rule to determine the inverse matrix for the following.

(a) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

4. Use Cramer's rule to solve the following for the unknown $\mathbf{x}$.

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. Use Cramer's rule to solve the following for the unknown $\mathbf{x}$.

(a) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

An *eigenvalue* λ for an $m \times m$ matrix A is a specific scalar value (i.e. a number) such that

$$\det(A - \lambda I) = 0.$$

For example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1.$$

Even for real matrices, its eigenvalues may be complex numbers. For example

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i.$$

When A is $m \times m$, refer back to the Leibniz formula for the determinant to see that $\det(A - \lambda I) = p_m(\lambda)$ where $p_m(\lambda)$ is a degree m polynomial in the variable λ. Let me show you why. Define $A_\lambda \equiv A - \lambda I$ and use Leibniz to write

$$\begin{aligned} \det(A_\lambda) &= \sum_{\mathbf{j} \in \mathcal{P}_m} \operatorname{sgn}(\mathbf{j})\, (A_\lambda)_{1,j_1} \cdots (A_\lambda)_{m,j_m} \\ &= \operatorname{sgn}(1,2,\ldots,m)(A_\lambda)_{1,1} \cdots (A_\lambda)_{m,m} + \sum_{\substack{\mathbf{j} \in \mathcal{P}_m \\ \mathbf{j} \neq (1,2,\ldots,m)}} \operatorname{sgn}(\mathbf{j})\, (A_\lambda)_{1,j_1} \cdots (A_\lambda)_{m,j_m} \\ &= (a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) + \sum_{\substack{\mathbf{j} \in \mathcal{P}_m \\ \mathbf{j} \neq (1,2,\ldots,m)}} \operatorname{sgn}(\mathbf{j})\, (A_\lambda)_{1,j_1} \cdots (A_\lambda)_{m,j_m}. \end{aligned}$$

For permutations $\mathbf{j} \neq (1,2,\ldots,m)$ we can have $j_i = i$ at most $m-2$ times. This says the product in the right hand sum can contain no more than $m-2$ diagonal terms of A_λ. Therefore

$$\det(A_\lambda) = (a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) + q_{m-2}(\lambda),$$

where $q_{m-2}(\lambda)$ is a polynomial which has degree no larger than $m-2$. Furthermore, it's easy to see

$$(a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) = (-1)^m \left(\lambda^m - (a_{1,1} + \cdots + a_{m,m})\, \lambda^{m-1} + \cdots \right)$$

and so insert this into above to conclude

$$\det(A - \lambda I) \equiv \det(A_\lambda) = \pm \left(\lambda^m - (a_{1,1} + \cdots + a_{m,m})\, \lambda^{m-1} \right) + \tilde{q}_{m-2}(\lambda),$$

where $\tilde{q}_{m-2}(\lambda)$ denotes some other polynomial with degree no larger than $m-2$.

$\det(A - \lambda I) \equiv p_m(\lambda)$ is called A's *characteristic polynomial.* To determine A's eigenvalues, we must therefore find all roots to its degree m characteristic polynomial.

When $m = 2$, you'll need to know the quadratic formula. For example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = \lambda^2 - 5\lambda - 2 = 0$$

$$\Rightarrow \quad \lambda = \frac{5 \pm \sqrt{33}}{2}.$$

When $m \geq 3$, you'll need to rely on me to give you a problem where you either "luck–out" and factor the characteristic polynomial directly or in the "worst–case" guess some of the characteristic roots in order to reduce to a quadratic by long division.

Here's a 3×3 example. Suppose

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix}.$$

Cofactor $A - \lambda I$ along, for example, the first row to get

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(-1)^{1+1} \det \begin{pmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{pmatrix} \\ &\qquad - 1(-1)^{1+2} \det \begin{pmatrix} -2 & 2 \\ -1 & 5-\lambda \end{pmatrix} + 1(-1)^{1+3} \det \begin{pmatrix} -2 & 4-\lambda \\ -1 & 1 \end{pmatrix} \\ &= (3-\lambda)\Big((4-\lambda)(5-\lambda) - 2\Big) + \Big(-2(5-\lambda)+2\Big) + \Big(-2+(4-\lambda)\Big) \\ &= (3-\lambda)\Big((4-\lambda)(5-\lambda) - 2\Big) + \Big(2\lambda - 8\Big) + \Big(2-\lambda\Big) \\ &= (3-\lambda)(4-\lambda)(5-\lambda) + 3(\lambda - 4). \end{aligned}$$

Yay! $(\lambda - 4)$ factors out. So,

$$\det(A - \lambda I) = -(\lambda-4)\Big((\lambda-3)(\lambda-5) - 3\Big) = -(\lambda-4)\Big(\lambda^2 - 8\lambda + 12\Big).$$

You can tell this is a homework problem because

$$\det(A - \lambda I) = -(\lambda-4)(\lambda-2)(\lambda-6) = 0 \quad \Rightarrow \quad \lambda = 2,\ 4,\ 6.$$

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6. Determine all eigenvalues for each of the following matrices.

$$\text{(a)} \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}$$

My answers: (a) $\lambda = 1, 2$. (b) $\lambda = 1, 2$. (c) $\lambda = 2, 3$. (d) $\lambda = 1, 2$.

7. Determine all eigenvalues for each of the following matrices.

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix}$$

My answers: (a) $\lambda = (3 \pm \sqrt{17})/2$. (b) $\lambda = 1 \pm i$. (c) $\lambda = 1, 2, 3$. (d) $\lambda = 1, 2, 3$.
