Dimension of a Vector Space

A vector space is said to have *finite dimension* if it can be spanned by a finite number of vectors.

A so-called *trivial vector space* is composed of a single vector $\mathbf{0}$. Its basis set is the empty set, and we'll say it is 0–dimensional.

Every finite dimensional vector space $\mathcal{V}$ has a finite basis. To see this is true, assume the vector space is nontrivial, otherwise the result is obvious. I will construct a basis set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ as follows.

Step 1: Select a nonzero vector $\mathbf{b}_1 \in \mathcal{V}$.

Step $k+1$: Suppose we have iteratively selected the set of vectors $B_k = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$. Now, select $\mathbf{b}_{k+1} \in \mathcal{V}$ satisfying $\mathbf{b}_{k+1} \notin \operatorname{span} B_k$. If there is such a vector, let $B_{k+1} = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k+1}\}$, increment k and repeat this step. If there is no such vector, terminate the iteration, and observe we have constructed a spanning set $\mathcal{V} = \operatorname{span} B_k$.

The algorithm will terminate in a finite number of steps, say $k = n$, and it yields a spanning set $B_n = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $\mathcal{V} = \operatorname{span} B_n$.

I claim B_n must also be independent. If not, there would be a collection of scalars, say $\alpha_1, \ldots, \alpha_n$, not all zero such that

$$\alpha_1 \mathbf{b}_1 + \cdots \alpha_n \mathbf{b}_n = \mathbf{0}.$$

Let $\tilde{n}$ be the largest index such that $\alpha_{\tilde{n}} \neq 0$. Then

$$\alpha_1 \mathbf{b}_1 + \cdots \alpha_{\tilde{n}} \mathbf{b}_{\tilde{n}} = \mathbf{0} \implies \mathbf{b}_{\tilde{n}} = -\frac{\alpha_1}{\alpha_{\tilde{n}}}\,\mathbf{b}_1 - \cdots - \frac{\alpha_{\tilde{n}-1}}{\alpha_{\tilde{n}}}\,\mathbf{b}_{\tilde{n}-1} \implies \mathbf{b}_{\tilde{n}} \in \operatorname{span} B_{\tilde{n}-1}.$$

But this contradicts the construction above which required $\mathbf{b}_{k+1} \notin \operatorname{span} B_k$ for each k. Therefore, conclude B_n is an independent spanning set, i.e. a basis.

Here's a simple yet surprisingly useful result. Suppose $B_n = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space $\mathcal{V}$. Consider any set of $n+1$ vectors from $\mathcal{V}$, say $X_{n+1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$. Then, the vectors in X_{n+1} must be linearly dependent.

To show this fact, let me use the Σ notation for vector sums in order to simplify notation. Since each $\mathbf{x}_k \in \mathcal{V}$, write these $n+1$ vectors in terms of the vectors in the basis set B_n,

$$\mathbf{x}_k = \sum_{i=1}^{n} a_{i,k}\,\mathbf{b}_i, \quad \text{for } k = 1, \ldots, n+1,$$

where $a_{i,k}$ are certain scalars which can be determined. Now, write

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_{n+1} \mathbf{x}_{n+1} = \mathbf{0} \implies \sum_{k=1}^{n+1} \alpha_k \sum_{i=1}^{n} a_{i,k}\,\mathbf{b}_i = \sum_{i=1}^{n} \left(\sum_{k=1}^{n+1} a_{i,k}\,\alpha_k \right) \mathbf{b}_i = \mathbf{0}.$$

Since the vectors in B_n are independent, this implies $\sum_{k=1}^{n+1} a_{i,k}\, \alpha_k = 0$ for each $i = 1, \ldots, n$. These n linear and homogeneous equations can be written as the following system of n equations in $n+1$ unknowns

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n+1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because the system has more unknowns than equations, when the associated augmented matrix is reduced to echelon form we must find at least one free variable (make sure you understand why). Free variables lead us to nontrivial solutions of the homogeneous problem. Therefore, $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_{n+1}\mathbf{x}_{n+1} = \mathbf{0}$ does <u>not</u> imply $\alpha_1 = \cdots = \alpha_{n+1} = 0$ which tells us the set X_{n+1} is in fact linearly dependent.

The *dimension* of a finite dimensional vector space $\mathcal{V}$ is given by the number of vectors in its basis set. But there could be a multitude of bases for $\mathcal{V}$. Does each always contain the same number of vectors? The answer is yes, and the next paragraph explains why.

Suppose $\mathcal{V}$ had two bases containing a different number of vectors. Call these B_1 and B_2. For the purpose of contradiction assume WLOG that B_2 has more vectors than B_1. Say B_1 contains n vectors and extract $n+1$ vectors from B_2. Apply the result above to conclude the extracted $n+1$ vectors must be linearly dependent. However, this is impossible because by supposition B_2 is a basis and is therefore composed of independent vectors.

The last thing I want to show you here is the following. If $\mathcal{S}$ is a subspace of a finite dimensional vector space $\mathcal{V}$ and $\dim(\mathcal{S}) = \dim(\mathcal{V})$, then $\mathcal{S} = \mathcal{V}$.

Here's the proof. Suppose $\dim(\mathcal{S}) = \dim(\mathcal{V}) = n$. For the purpose of contradiction, assume $\mathcal{S} \neq \mathcal{V}$. $\mathcal{S} \subseteq \mathcal{V}$ together with the assumption $\mathcal{S} \neq \mathcal{V}$ says there is a vector $\mathbf{x} \in \mathcal{V}$ but $\mathbf{x} \notin \mathcal{S}$. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $\mathcal{S}$, and consider the $n+1$ vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{x}$ which are all contained in $\mathcal{V}$. The fact that $\mathbf{x} \notin \mathcal{S}$ implies these $n+1$ vectors form an independent set. However, our useful result given earlier implies $n+1$ vectors embedded within an n-dimensional vector space must always be dependent. These two facts can't both be true, and so by contradiction we conclude that $\mathcal{S} = \mathcal{V}$.