Eigenvalues and Eigenvectors

The *eigenvalues* of an $m \times m$ square matrix $A$ are the roots of its degree $m$ *characteristic polynomial*, $p(\lambda) \equiv \det(A - \lambda I)$. Eigenvalues may be real numbers but they can in general be complex numbers.

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A few of these exercises are also on your previous homework.

1. Determine the characteristic polynomial and then compute the eigenvalues for the following matrices.

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Answers: (a) $\lambda_1 = -1$, $\lambda_2 = 3$, (b) $\lambda_1 = \lambda_2 = 1$ (this eigenvalue has multiplicity 2), (c) $\lambda_1 = 1$, $\lambda_2 = 2$, (d) $\lambda_1 = 1 - i$, $\lambda_2 = 1 + i$.

2. Do the same as in the previous exercise for the following.

$$\text{(a)} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix}$$

Answers: (a) $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 3$, (b) $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$.

3. Do the same for these.

$$\text{(a)} \begin{pmatrix} 3 & 0 & -1 & 1 \\ 1 & 8 & 2 & 3 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 1 & 5 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

Answers: (a) $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$, $\lambda_4 = 8$, (b) $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = \lambda_4 = 3$.

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Each distinct eigenvalue, say $\lambda_i$, has associated to it at least one *eigenvector*, say $\mathbf{r}_{\lambda_i}$. An eigenvector is a nonzero vector satisfying

$$A\mathbf{r}_{\lambda_i} = \lambda_i \mathbf{r}_{\lambda_i}.$$

Once the eigenvalues are determined by factoring the characteristic polynomial, the eigenvectors associated to each distinct eigenvalue $\lambda_i$ are determined by finding a basis for

$$\mathcal{E}_{\lambda_i} \equiv \text{Null}(A - \lambda_i I).$$

This *eigenspace* $\mathcal{E}_{\lambda_i}$ is always at least one dimensional. However, if $\lambda_i$ has multiplicity greater than one it is possible for $\dim \mathcal{E}_{\lambda_i} > 1$. In general, it can be shown that

$$1 \leq \dim \mathcal{E}_{\lambda_i} \leq m_{\lambda_i},$$

where $m_{\lambda_i}$ is the algebraic multiplicity of the characteristic root (eigenvalue) $\lambda_i$.

Here's an important result you should all know. Suppose a matrix $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e. $\lambda_i \neq \lambda_j$ for all $i \neq j$, and suppose $\mathbf{r}_1, \ldots, \mathbf{r}_n$ denotes their associated eigenvectors, i.e. $A\mathbf{r}_i = \lambda_i \mathbf{r}_i$ with $\mathbf{r}_i \neq \mathbf{0}$ for all $i$. Because the eigenvalues are distinct, it follows that $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ is necessarily a linearly independent set. I'll use induction to show this fact is true.

Clearly the set $\{\mathbf{r}_1\}$ with one nonzero vector is linearly independent. For the purpose of induction, assume for arbitrary $1 \leq k < n$ the set $\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}$ is independent. Now I need to show this assumption implies the set $\{\mathbf{r}_1, \ldots, \mathbf{r}_k, \mathbf{r}_{k+1}\}$ is also independent. With this goal in mind, suppose there are scalars such that

$$\alpha_1 \mathbf{r}_1 + \cdots + \alpha_k \mathbf{r}_k + \alpha_{k+1} \mathbf{r}_{k+1} = \mathbf{0}.$$

Multiply by $\lambda_{k+1}$ and then apply the matrix $A$ to the sum above to get

$$\begin{aligned} \alpha_1 \lambda_{k+1} \mathbf{r}_1 + \cdots + \alpha_k \lambda_{k+1} \mathbf{r}_k + \alpha_{k+1} \lambda_{k+1} \mathbf{r}_{k+1} &= \mathbf{0}, \\ \alpha_1 \lambda_1 \mathbf{r}_1 + \cdots + \alpha_k \lambda_k \mathbf{r}_k + \alpha_{k+1} \lambda_{k+1} \mathbf{r}_{k+1} &= \mathbf{0}. \end{aligned}$$

Subtract the second from the first

$$\alpha_1 \left(\lambda_{k+1} - \lambda_1\right) \mathbf{r}_1 + \cdots + \alpha_k \left(\lambda_{k+1} - \lambda_k\right) \mathbf{r}_k = \mathbf{0}.$$

But the assumption that $\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}$ is independent implies for each $1 \leq i \leq k$ we must have $\alpha_i \left(\lambda_{k+1} - \lambda_i\right) = 0$, and $\lambda_{k+1} - \lambda_i \neq 0$ implies $\alpha_i = 0$. The scalar $\alpha_{k+1}$ must also be zero because the eigenvector $\mathbf{r}_{k+1} \neq \mathbf{0}$. Therefore we have shown $\{\mathbf{r}_1, \ldots, \mathbf{r}_k, \mathbf{r}_{k+1}\}$ is an independent set. Induction now allows us to conclude this is true for any $1 \leq k < n$.

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4. Determine all eigenvectors for the matrices given in exercise 1.

5. Determine all eigenvectors for the matrices given in exercise 2.

6. Determine all eigenvectors for the matrices given in exercise 3.

Answers for exercises 4–6.

Matrices from 1.

(a) $\lambda = -1,\ \mathbf{r} = (1, -1)^T, \quad \lambda = 3,\ \mathbf{r} = (1, 1)^T.$

(b) $\lambda = 1,\ \mathbf{r} = (1, 0)^T.$

(c) $\lambda = 1,\ \mathbf{r} = (3, 1)^T, \quad \lambda = 2,\ \mathbf{r} = (2, 1)^T.$

(d) $\lambda = 1 - i,\ \mathbf{r} = (1, i)^T, \quad \lambda = 1 + i,\ \mathbf{r} = (1, -i)^T.$

Matrices from 2.

(a) $\lambda = -1,\ \mathbf{r} = (1, -1, 0)^T, \quad \lambda = 3,\ \mathbf{r} = (1, 1, 0)^T.$

(b) $\lambda = 2,\ \mathbf{r} = (1, 1, 0)^T, \quad \lambda = 4,\ \mathbf{r} = (1, 0, 1)^T, \quad \lambda = 6,\ \mathbf{r} = (0, 1, 1)^T.$

Matrices from 3.

(a) $\lambda = 2,\ \mathbf{r} = (2, -1, 2, 0)^T, \quad \lambda = 4,\ \mathbf{r} = (1, -1, 0, 1)^T,$

$\lambda = 6,\ \mathbf{r} = (0, -5, 2, 2)^T, \quad \lambda = 8,\ \mathbf{r} = (0, 1, 0, 0)^T.$

(b) $\lambda = -1,\ \mathbf{r} = (1, -1, 0, 0)^T,\ (0, 0, 1, -1)^T, \quad \lambda = 3,\ \mathbf{r} = (1, 1, 0, 0)^T,\ (0, 0, 1, 1)^T.$

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