Eigenvalues and Eigenvectors

The eigenvalues of an $m \times m$ square matrix $A$ are the roots of its degree $m$ characteristic polynomial, $p(\lambda) \equiv \det(A - \lambda I)$. Eigenvalues may be real numbers but they can in general be complex numbers.

A few of these exercises are also on your previous homework.

1. Determine the characteristic polynomial and then compute the eigenvalues for the following matrices.

(a) \[
\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}
\]  
(b) \[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]  
(c) \[
\begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}
\]  
(d) \[
\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

Answers: (a) $\lambda_1 = -1$, $\lambda_2 = 3$, (b) $\lambda_1 = \lambda_2 = 1$ (this eigenvalue has multiplicity 2), (c) $\lambda_1 = 1$, $\lambda_2 = 2$, (d) $\lambda_1 = 1 - i$, $\lambda_2 = 1 + i$.

2. Do the same as in the previous exercise for the following.

(a) \[
\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}
\]  
(b) \[
\begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix}
\]

Answers: (a) $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 3$, (b) $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$.

3. Do the same for these.

(a) \[
\begin{pmatrix} 3 & 0 & -1 & 1 \\ 1 & 8 & 2 & 3 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 1 & 5 \end{pmatrix}
\]  
(b) \[
\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}
\]

Answers: (a) $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$, $\lambda_4 = 8$, (b) $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = \lambda_4 = 3$.

Each distinct eigenvalue, say $\lambda_i$, has associated to it at least one eigenvector, say $r_{\lambda_i}$. An eigenvector is a nonzero vector satisfying

$$Ar_{\lambda_i} = \lambda_i r_{\lambda_i}.$$  

Once the eigenvalues are determined by factoring the characteristic polynomial, the eigenvectors associated to each distinct eigenvalue $\lambda_i$ are determined by finding a basis for

$$E_{\lambda_i} \equiv \text{Null}(A - \lambda_i I).$$

This eigenspace $E_{\lambda_i}$ is always at least one dimensional. However, if $\lambda_i$ has multiplicity greater than one it is possible for $\dim E_{\lambda_i} > 1$. In general, it can be shown that

$$1 \leq \dim E_{\lambda_i} \leq m_{\lambda_i},$$

where $m_{\lambda_i}$ is the algebraic multiplicity of the characteristic root (eigenvalue) $\lambda_i$. 

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4. Determine all eigenvectors for the matrices given in exercise 1.

5. Determine all eigenvectors for the matrices given in exercise 2.

6. Determine all eigenvectors for the matrices given in exercise 3.

Answers for exercises 4–6.

Matrices from 1.

(a) \( \lambda = -1, \ r = (1, -1)^T, \ \lambda = 3, \ r = (1, 1)^T \).

(b) \( \lambda = 1, \ r = (1, 0)^T \).

(c) \( \lambda = 1, \ r = (3, 1)^T, \ \lambda = 2, \ r = (2, 1)^T \).

(d) \( \lambda = 1 - i, \ r = (1, i)^T, \ \lambda = 1 + i, \ r = (1, -i)^T \).

Matrices from 2.

(a) \( \lambda = -1, \ r = (1, -1, 0)^T, \ \lambda = 3, \ r = (1, 1, 0)^T \).

(b) \( \lambda = 2, \ r = (1, 1, 0)^T, \ \lambda = 4, \ r = (1, 0, 1)^T, \ \lambda = 6, \ r = (0, 1, 1)^T \).

Matrices from 3.

(a) \( \lambda = 2, \ r = (2, -1, 2, 0)^T, \ \lambda = 4, \ r = (1, -1, 0, 1)^T, \lambda = 6, \ r = (0, -5, 2, 2)^T, \ \lambda = 8, \ r = (0, 1, 0, 0)^T \).

(b) \( \lambda = -1, \ r = (1, -1, 0, 0)^T, (0, 0, 1, -1)^T, \ \lambda = 3, \ r = (1, 1, 0, 0)^T, (0, 0, 1, 1)^T \).