Linear Operators

Let $\mathcal{V}_x$ and $\mathcal{V}_y$ denote two finite dimensional vector spaces who both share the same scalar field. Let $\mathcal{L}$ denote a map from $\mathcal{V}_x$ to $\mathcal{V}_y$ signified by the notation $\mathcal{L} : \mathcal{V}_x \to \mathcal{V}_y$. That is, for every $\mathbf{x} \in \mathcal{V}_x$ we have $\mathcal{L}(\mathbf{x})$ is a vector in $\mathcal{V}_y$. $\mathcal{L}$ is said to be a *linear operator* if for all vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x} \in \mathcal{V}_x$ and scalars α we have

$$\mathcal{L}(\mathbf{x}_1 + \mathbf{x}_2) = \mathcal{L}(\mathbf{x}_1) + \mathcal{L}(\mathbf{x}_2) \quad \text{and} \quad \mathcal{L}(\alpha\mathbf{x}) = \alpha\mathcal{L}(\mathbf{x}).$$

Here're two examples.

A 3×2 real matrix A defines a linear operator from $\mathbb{R}^2$ into $\mathbb{R}^3$. Clearly the product $A\mathbf{x}$ is defined for any $\mathbf{x} \in \mathbb{R}^2$ and the result is a vector in $\mathbb{R}^3$. To see $\mathcal{L}(\mathbf{x}) \equiv A\mathbf{x}$ is linear, the matrix distributive property $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$ and scalar/matrix multiplication $A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$ should make this fact clear.

Let $\mathcal{P}_n$ denote the space of real polynomials of degree less than or equal to n. The derivative d/dx is a linear operator from $\mathcal{P}_n$ into $\mathcal{P}_n$. Clearly for any $p \in \mathcal{P}_n$, $dp/dx \in \mathcal{P}_{n-1} \subseteq \mathcal{P}_n$. To see this is linear, recall from calculus that the derivative of a sum is the sum of the derivatives, $d(p+q)/dx = dp/dx + dq/dx$, and constants can be pulled through the derivative, $d(cp)/dx = c\, dp/dx$.

There are two important subspace associated to a linear $\mathcal{L} : \mathcal{V}_x \to \mathcal{V}_y$. First, there is the *null space* of $\mathcal{L}$

$$\mathrm{Null}(\mathcal{L}) \equiv \{\mathbf{x} \in \mathcal{V}_x : \mathcal{L}(\mathbf{x}) = \mathbf{0}\} \subseteq \mathcal{V}_x.$$

Second is the *range space* of $\mathcal{L}$

$$\mathrm{Rang}(\mathcal{L}) \equiv \{\mathbf{y} = \mathcal{L}(\mathbf{x}) : \mathbf{x} \in \mathcal{V}_x\} \subseteq \mathcal{V}_y.$$

Think of the null space of $\mathcal{L}$ as the set of all vectors $\mathbf{x} \in \mathcal{V}_x$ that are annihilated by $\mathcal{L}$. You are asked in an exercise below to show $\mathrm{Null}(\mathcal{L})$ is in fact a <u>subspace</u> of $\mathcal{V}_x$.

Think of the range space of $\mathcal{L}$ as the set of all vectors $\mathbf{y} \in \mathcal{V}_y$ that can be reached via $\mathcal{L}$ for some vector in $\mathbf{x} \in \mathcal{V}_x$. That is $\mathbf{y} = \mathcal{L}(\mathbf{x})$ for some $\mathbf{x} \in \mathcal{V}_x$. You are asked in an exercise to also show $\mathrm{Rang}(\mathcal{L})$ is a <u>subspace</u> of $\mathcal{V}_y$.

Here are two terms often used in linear algebra. The *nullity* of a linear operator is given by the dimension of its null space. The *rank* of a linear operator is given by the dimension of its range space. You'll hear the term rank often in this course.

A fundamental theorem from linear algebra is called the rank theorem. Its statement is:

$$\text{Given linear } \mathcal{L} : \mathcal{V}_x \to \mathcal{V}_y \quad \Rightarrow \quad \dim(\mathrm{Null}(\mathcal{L})) + \dim(\mathrm{Rang}(\mathcal{L})) = \dim(\mathcal{V}_x).$$

I'll give an elementary proof of this important fact in a subsequent set of notes.

Let me give a couple of examples.

First, let's consider the 2×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{pmatrix}$$

which defines a linear operator $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^2$. To compute $\mathcal{L}$'s null space, find all solutions to $A\mathbf{x} = \mathbf{0}$ by elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{aligned} x_3 &= 0 \\ x_2 &= \alpha \\ x_1 &= -2\alpha. \end{aligned}$$

This tells us

$$\text{Null}(\mathcal{L}) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \Rightarrow \quad \dim(\text{Null}(\mathcal{L})) = 1.$$

Next, to compute $\mathcal{L}$'s range space, observe

$$A\mathbf{x} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad \text{Rang}(\mathcal{L}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}.$$

As I warned you on an earlier assignment, make sure you understand what I did above, i.e. I wrote

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In general, you should all see that the range of a matrix is given by the span of its column vectors. Make sure to get this! The range's spanning set given above is not linearly independent. However, we can compute its standard basis, see homework titled "Introduction to Vector Spaces" exercise 10, to get

$$\text{Rang}(\mathcal{L}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \dim(\text{Rang}(\mathcal{L})) = 2.$$

This of course is in agreement with what the rank theorem says

$$\dim(\text{Null}(\mathcal{L})) + \dim(\text{Rang}(\mathcal{L})) = \dim(\mathcal{V}_x) \quad \Rightarrow \quad 1 + 2 = \dim(\mathbb{R}^3) = 3.$$

Second, let's consider

$$\mathcal{L}(p) \equiv \frac{dp}{dx} + p \;\text{ as a linear operator from } \mathcal{P}_2 \text{ into } \mathcal{P}_2.$$

Let $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ be an arbitrary vector from $\mathcal{P}_2$ and calculate

$$0 = \frac{dp}{dx} + p = (\alpha_1 + 2\alpha_2 x) + (\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_1 + \alpha_0) + (2\alpha_2 + \alpha_1)\, x + \alpha_2 x^2.$$

But this implies $\alpha_2 = 0$ which implies $\alpha_1 = 0$ which implies $\alpha_0 = 0$. Therefore

$$\mathrm{Null}(\mathcal{L}) = \mathrm{span}\,\{\mathbf{0}\} \quad \Rightarrow \quad \dim(\mathrm{Null}(\mathcal{L})) = 0.$$

Analogous to the column span we used for the 2×3 matrix above, see that

$$\begin{aligned} &\mathcal{L}(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = \alpha_0\mathcal{L}(1) + \alpha_1\mathcal{L}(x) + \alpha_2\mathcal{L}(x^2) \\ \Rightarrow \quad &\mathrm{Rang}(\mathcal{L}) = \mathrm{span}\{\mathcal{L}(1),\, \mathcal{L}(x),\, \mathcal{L}(x^2)\} = \mathrm{span}\{1,\, 1+x,\, 2x+x^2\}. \end{aligned}$$

You check that the spanning set above is linearly independent. Therefore, by counting the number of basis vectors, $\dim(\mathrm{Rang}(\mathcal{L})) = 3$ which also implies $\mathrm{Rang}(\mathcal{L}) = \mathcal{P}_2$.

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1. Suppose $\mathcal{L} : \mathcal{V}_x \to \mathcal{V}_y$ is linear. (a) Prove $\mathrm{Null}(\mathcal{L})$ is a subspace of $\mathcal{V}_x$. (b) Prove $\mathrm{Rang}(\mathcal{L})$ is a subspace of $\mathcal{V}_y$.

2. Suppose $\mathcal{V}$ is a finite dimensional vector space and $\mathcal{L} : \mathcal{V} \to \mathcal{V}$ is linear. Also, suppose $\mathrm{Null}(\mathcal{L}) = \{\mathbf{0}\}$. Prove that for any vector $\mathbf{y} \in \mathcal{V}$ there exists a unique vector $\mathbf{x} \in \mathcal{V}$ such that $\mathcal{L}(\mathbf{x}) = \mathbf{y}$.

Each of the following matrices define a linear operator from $\mathbb{R}^3$ into $\mathbb{R}^3$. (a) Determine a basis for the null space. (b) Determine the <u>standard</u> basis for the range space.

3. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ 4. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix}$

For 3 I got: (a) $\{(1,0,-1)^T\}$, (b) $\{(1,0,-1)^T, (0,1,1)^T\}$.
For 4 I got: (a) $\{(1,0,-1)^T, (-2,1,0)^T\}$, (b) $\{(1,2,3)^T\}$.

Each of the following matrices define a linear operator from $\mathbb{R}^3$ into $\mathbb{R}^4$ in exercise 5, $\mathbb{R}^3$ into $\mathbb{R}^3$ in exercise 6 and $\mathbb{R}^4$ into $\mathbb{R}^3$ in exercise 7. (a) Determine a basis for the null space. (b) Determine the <u>standard</u> basis for the range space. (c) Compute the dimensions of the null and range spaces in order to confirm the result of the rank theorem.

5. $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 4 & 9 \\ 4 & 2 & 2 \end{pmatrix}$ 6. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ 7. $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

I got for 5(a) $\{(1,-3,1)^T\}$, 6(a) $\{(0,1,-1)^T\}$, 7(a) $\{(1,-4,-1,2)^T\}$.

8. Let $\mathcal{P}_2$ denote the space of real polynomials having degree less than or equal to two. Consider the linear operator $\mathcal{L}(p) \equiv d^2p/dx^2 + dp/dx$ that maps $\mathcal{P}_2$ into $\mathcal{P}_2$. (a) Determine a basis for $\mathcal{L}$'s null space. (b) Determine a basis for $\mathcal{L}$'s range space.
I got: (a) $\{1\}$, (b) $\{1,\, 1+x\}$. BTW. $\{1,\, x\}$ is another basis for $\mathcal{L}$'s range space.
