Proof of the Rank Theorem

The rank theorem was stated in your earlier "Linear Operators" homework assignment. It reads as follows. Let $\mathcal{V}_x$ and $\mathcal{V}_y$ denote two vector spaces which share scalars, suppose $\mathcal{V}_x$ is finite dimensional and $\mathcal{L} : \mathcal{V}_x \to \mathcal{V}_y$ is linear. Then

$$\dim(\mathrm{Null}(\mathcal{L})) + \dim(\mathrm{Rang}(\mathcal{L})) = \dim(\mathcal{V}_x).$$

Before giving a proof of the rank theorem, let me give a definition and some related facts.

Suppose $\mathcal{M}$ and $\mathcal{N}$ are subspaces of a vector space $\mathcal{V}$ which only have the zero vector in common, i.e. $\mathcal{M} \cap \mathcal{N} = \{\mathbf{0}\}$. The *direct sum* of $\mathcal{M}$ and $\mathcal{N}$, denoted by $\mathcal{M} \oplus \mathcal{N}$, is a subspace of $\mathcal{V}$ whose vectors are given by

$$\mathcal{M} \oplus \mathcal{N} \equiv \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathcal{M},\ \mathbf{y} \in \mathcal{N}\}.$$

Please note, the direct sum notation, $\oplus$ in $\mathcal{M} \oplus \mathcal{N}$, tacitly implies $\mathcal{M} \cap \mathcal{N} = \{\mathbf{0}\}$. Here's what you want to think of when faced with the direct sum of finite dimensional vector spaces. Given basis sets for $\mathcal{M}$ and $\mathcal{N}$

$$\{\mathbf{m}_1, \ldots, \mathbf{m}_{d_m}\} \text{ and } \{\mathbf{n}_1, \ldots, \mathbf{n}_{d_n}\} \quad \Rightarrow \quad \{\mathbf{m}_1, \ldots, \mathbf{m}_{d_m}, \mathbf{n}_1, \ldots, \mathbf{n}_{d_n}\}$$

is a basis for $\mathcal{M} \oplus \mathcal{N}$. Clearly the set on the right is a spanning set for $\mathcal{M} \oplus \mathcal{N}$. Moreover, since $\mathcal{M} \cap \mathcal{N} = \{\mathbf{0}\}$, the set on the right is easily seen to be an <u>independent</u> spanning set. Now, conclude that

$$\dim(\mathcal{M} \oplus \mathcal{N}) = \dim(\mathcal{M}) + \dim(\mathcal{N}),$$

and observe we have the unique decomposition

$$\forall \mathbf{z} \in \mathcal{M} \oplus \mathcal{N} \quad \Rightarrow \quad \exists!\ \mathbf{m} \in \mathcal{M},\ \mathbf{n} \in \mathcal{N} \text{ such that } \mathbf{z} = \mathbf{m} + \mathbf{n}.$$

The symbols $\forall$ reads "for all" and $\exists!$ reads "there exists unique".

We'll need one more small preliminary result. Given a finite dimensional vector space $\mathcal{V}$ and a subspace $\mathcal{N} \subseteq \mathcal{V}$, there is a subspace $\mathcal{M} \subseteq \mathcal{V}$ such that $\mathcal{V} = \mathcal{N} \oplus \mathcal{M}$. $\mathcal{M}$ can be constructed by an algorithm essentially identical to the one given near the top of your "Dimension of a Vector Space" notes used there to construct a basis for any finite dimensional vector space. I'm going to omit the details here.

Here's a proof of the rank theorem. Decompose $\mathcal{V}_x$ as

$$\mathcal{V}_x = \mathrm{Null}(\mathcal{L}) \oplus \mathcal{M} \text{ where from above } \mathcal{M} \text{ has a basis } M = \{\mathbf{m}_1, \ldots, \mathbf{m}_{d_m}\}.$$

Since $\dim(\mathcal{V}_x) = \dim(\mathrm{Null}(\mathcal{L})) + \dim(\mathcal{M})$ we see $d_m = \dim(\mathcal{M}) = \dim(\mathcal{V}_x) - \dim(\mathrm{Null}(\mathcal{L}))$. Now consider the set of vectors each coming from $\mathrm{Rang}(\mathcal{L}) \subseteq \mathcal{V}_y$

$$L = \{\mathcal{L}(\mathbf{m}_1), \ldots, \mathcal{L}(\mathbf{m}_{d_m})\}.$$

Clearly $\operatorname{span} L \subseteq \operatorname{Rang}(\mathcal{L})$. Moreover, for arbitrary $\mathbf{x} \in \mathcal{V}_x$ we can decompose $\mathbf{x} = \mathbf{n} + \mathbf{m}$ where $\mathbf{n} \in \operatorname{Null}(\mathcal{L})$ and $\mathbf{m} \in \mathcal{M}$, and from this conclude

$$\mathcal{L}(\mathbf{x}) = \mathcal{L}(\mathbf{n}) + \mathcal{L}(\mathbf{m}) = \mathbf{0} + \mathcal{L}(\mathbf{m}) \in \operatorname{span} L \quad \Rightarrow \quad \operatorname{Rang}(\mathcal{L}) \subseteq \operatorname{span} L.$$

Therefore, $\operatorname{span} L = \operatorname{Rang}(\mathcal{L})$. Finally, let's check that L is independent. Suppose

$$\begin{aligned}
& \mathbf{0} = \alpha_1 \mathcal{L}(\mathbf{m}_1) + \cdots + \alpha_{d_m} \mathcal{L}(\mathbf{m}_{d_m}) = \mathcal{L}(\alpha_1 \mathbf{m}_1 + \cdots + \alpha_{d_m} \mathbf{m}_{d_m}) \\
\Rightarrow \quad & \alpha_1 \mathbf{m}_1 + \cdots + \alpha_{d_m} \mathbf{m}_{d_m} \in \operatorname{Null}(\mathcal{L}) \\
\Rightarrow \quad & \alpha_1 \mathbf{m}_1 + \cdots + \alpha_{d_m} \mathbf{m}_{d_m} = \mathbf{0} \;\; \text{because } \operatorname{Null}(\mathcal{L}) \cap \mathcal{M} = \{\mathbf{0}\} \\
\Rightarrow \quad & \alpha_1 = \cdots = \alpha_{d_m} = 0 \qquad\qquad\;\; \text{because } M \text{ is a basis for } \mathcal{M},
\end{aligned}$$

and so L is an independent spanning set of $\operatorname{Rang}(\mathcal{L})$. Count vectors in L to get

$$\begin{aligned}
& \dim(\operatorname{Rang}(\mathcal{L})) = \dim(\operatorname{span} L) = d_m = \dim(\mathcal{V}_x) - \dim(\operatorname{Null}(\mathcal{L})) \\
\Rightarrow \quad & \dim(\operatorname{Null}(\mathcal{L})) + \dim(\operatorname{Rang}(\mathcal{L})) = \dim(\mathcal{V}_x)
\end{aligned}$$

which is the result of the rank theorem.