Two Vector Spaces That Are Not \mathbb{R}^m

Consider a vector space with the same scalars and vectors as in \mathbb{R}^2

scalars:
$$\alpha \in \mathbb{R}$$
 and vectors: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$

but with scalar multiplication and vector addition defined differently

$$\alpha \mathbf{x} \equiv \begin{pmatrix} \alpha(x_1 - 1) + 1 \\ \alpha(x_2 - 1) + 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} + \mathbf{y} \equiv \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix}.$$

Let's refer to this *strange* vector space as S^2 . Of course, in order to call S^2 a vector space, it must be verified that structural conditions (a–0) through (d–2) listed in your previous homework are in fact true. Clearly, (a–0) is true. Moreover, vector addition is associative and commutative

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 1 + z_1 - 1 \\ x_2 + y_2 - 1 + z_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 2 \\ x_2 + y_2 + z_2 - 2 \end{pmatrix},$$

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 + z_1 - 1 \\ y_2 + z_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 1 - 1 \\ x_2 + y_2 + z_2 - 1 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 2 \\ x_2 + y_2 + z_2 - 2 \end{pmatrix},$$

$$\mathbf{y} + \mathbf{x} = \begin{pmatrix} y_1 + x_1 - 1 \\ y_2 + x_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} = \mathbf{x} + \mathbf{y}.$$

You'll be asked to verify (a–3) and (a–4) in an exercise below. FYI:

the additive identity $\mathbf{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and \mathbf{x} 's additive inverse $\mathbf{x}' = \begin{pmatrix} 2 - x_1 \\ 2 - x_2 \end{pmatrix}$.

Continuing with the listed items, (m-0) is clearly true. For (m-1)

$$\alpha(\beta \mathbf{x}) = \alpha \begin{pmatrix} \beta(x_1 - 1) + 1\\ \beta(x_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} \alpha(\beta(x_1 - 1) + 1 - 1) + 1\\ \alpha(\beta(x_2 - 1) + 1 - 1) + 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha\beta(x_1 - 1) + 1\\ \alpha\beta(x_2 - 1) + 1 \end{pmatrix} = (\alpha\beta)\mathbf{x},$$

and (m-2)

$$1\mathbf{x} = \begin{pmatrix} 1(x_1 - 1) + 1\\ 1(x_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} x_1\\ x_2 \end{pmatrix} = \mathbf{x}.$$

For (d-1)

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} = \begin{pmatrix} \alpha(x_1 + y_1 - 1 - 1) + 1 \\ \alpha(x_2 + y_2 - 1 - 1) + 1 \end{pmatrix}$$

$$\begin{aligned} \alpha \mathbf{x} + \alpha \mathbf{y} &= \begin{pmatrix} \alpha(x_1 - 1) + 1\\ \alpha(x_2 - 1) + 1 \end{pmatrix} + \begin{pmatrix} \alpha(y_1 - 1) + 1\\ \alpha(y_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} \alpha(x_1 - 1) + 1 + \alpha(y_1 - 1) + 1 - 1\\ \alpha(x_2 - 1) + 1 + \alpha(y_2 - 1) + 1 - 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha(x_1 + y_1 - 1 - 1) + 1\\ \alpha(x_2 + y_2 - 1 - 1) + 1 \end{pmatrix},\end{aligned}$$

and so $\alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$. You'll be asked to verify property (d-2) in an exercise below.

1. Let \mathcal{V} be any vector space.

(a) Suppose **0** and **0'** are additive identities for \mathcal{V} . Show **0'** = **0** to conclude \mathcal{V} 's additive identity is unique.

(b) Let 0 denote \mathcal{V} 's scalar field additive identity and let **0** denote \mathcal{V} 's vector additive identity. For any $\mathbf{x} \in \mathcal{V}$ show that $0\mathbf{x} = \mathbf{0}$.

(c) Suppose $\mathbf{x} \in \mathcal{V}$ has additive inverses \mathbf{x}' and \mathbf{x}'' . Show $\mathbf{x}'' = \mathbf{x}'$ to conclude \mathbf{x} 's additive inverse is unique.

(d) Let -1 denote \mathcal{V} 's scalar field additive inverse of its multiplicative identity 1. For any $\mathbf{x} \in \mathcal{V}$ show that $-1\mathbf{x} = \mathbf{x}'$ where \mathbf{x}' is \mathbf{x} 's additive inverse.

Please justify each step by stating which properties from (a–0) through (d–2) were used.

2. Recall what I called S^2 above.

(a) Explicitly calculate the additive identity vector, $\mathbf{0}$, and the additive inverse of \mathbf{x} , \mathbf{x}' , in order to establish that S^2 satisfies vector space properties (a–3) and (a–4).

(b) Show S^2 satisfies property (d-2).

(c) For S^2 confirm by calculating that $0\mathbf{x} = \mathbf{0}$ and $-1\mathbf{x} = \mathbf{x}'$.

3. Consider the following three vectors in our strange vector space S^2 :

$$\mathbf{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Recall from the previous exercise that **0** here is S^2 's additive identity.

(a) Show on S^2 that the two vectors $\{\mathbf{x}, \mathbf{y}\}$ forms an independent set. Hint: You must conclude $\alpha \mathbf{x} + \beta \mathbf{y} = \mathbf{0} \iff \alpha = \beta = 0$. Remember how scalar multiplication and vector addition are defined on S^2 .

(b) Show that span{ \mathbf{x}, \mathbf{y} } = S^2 . Hint: For any $\mathbf{z} \in S^2$ I computed that

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
, $\alpha \mathbf{x} + \beta \mathbf{y} = \mathbf{z}$ where $\alpha = z_1 - 3z_2 + 2$ and $\beta = z_2 - 1$.

4. Now consider the following two vectors in S^2 :

$$\mathbf{x} = \begin{pmatrix} 2\\1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3\\1 \end{pmatrix}.$$

(a) Show that $\{\mathbf{x}, \mathbf{y}\}$ forms a dependent set on S^2 . Hint: I used $2\mathbf{x} = \mathbf{y}$.

(b) Describe a one parameter family of vectors which gives span{ \mathbf{x}, \mathbf{y} }. Hint: I used the fact that $\alpha \mathbf{x} + \beta \mathbf{y} = \alpha \mathbf{x} + \beta 2\mathbf{x} = (\alpha + 2\beta)\mathbf{x} = \tilde{\alpha}\mathbf{x}$.

Let \mathcal{P}_n denote the set of real polynomials of degree less than or equal to n; $n \ge 0$. Let scalars be given by real constants. Define scalar multiplication by pointwise multiplication of functions by constants

$$\alpha \in \mathbb{R}, \ \mathbf{p} \in \mathcal{P}_n, \ \alpha \mathbf{p} \equiv \alpha p(x),$$

and vector addition by pointwise addition of functions

$$\mathbf{p} \in \mathcal{P}_n$$
 and $\mathbf{q} \in \mathcal{P}_n$, $\mathbf{p} + \mathbf{q} \equiv p(x) + q(x)$.

The vector additive identity is the zero function. Constant functions including the zero constant function are taken to be polynomials of degree zero. With this, it's a routine exercise to show \mathcal{P}_n is a vector space.

Clearly, the set of polynomials $\{1, x, \ldots, x^n\}$ is a spanning set of vectors for \mathcal{P}_n . That is

$$\mathbf{p} \in \mathcal{P}_n \quad \Rightarrow \quad \mathbf{p} = p(x) = \alpha_0 1 + \alpha_1 x + \dots + \alpha_n x^n$$

for certain scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$. You'll be asked in exercise 5 below to show

$$\alpha_0 1 + \alpha_1 x + \dots + \alpha_n x^n = 0$$
 for all $x \iff \alpha_0 = \alpha_1 = \dots = \alpha_n = 0$

which says $\{1, x, \ldots, x^n\}$ is an independent set. Therefore, $\{1, x, \ldots, x^n\}$ is a basis (and in some sense the standard basis) for \mathcal{P}_n . Moreover, the dimension of \mathcal{P}_n is dim $(\mathcal{P}_n) = n+1$. Here's another basis for the space \mathcal{P}_2 . I'm taking n = 2 in order to make the present example easier to read. Consider

$$B = \{ (x-1)(x-2), (x-0)(x-2), (x-0)(x-1) \}.$$

FYI: This is the Lagrange basis for polynomial interpolation at x = 0, 1, 2. One easily sees span B is a subspace of \mathcal{P}_2 . Moreover, B is an independent set. To see this, assume

$$b(x) = \alpha_0 (x-1)(x-2) + \alpha_1 (x-0)(x-2) + \alpha_2 (x-0)(x-1) = \mathbf{0} = 0 \text{ for all } x.$$

But b(x) = 0 for all x implies in particular b(0) = b(1) = b(2) = 0. However, b(0) = 0 implies $2\alpha_0 = 0$, and b(1) = 0 implies $-\alpha_1 = 0$, and b(2) = 0 implies $2\alpha_2 = 0$. Therefore $\alpha_0 = \alpha_1 = \alpha_2 = 0$ which says B is independent. Let me again state (but prove later) the following general vector space result.

If
$$\mathcal{S} \subseteq \mathcal{V}$$
 and $\dim(\mathcal{S}) = \dim(\mathcal{V})$ then $\mathcal{S} = \mathcal{V}$.

Returning to our example, since span $B \subseteq \mathcal{P}_2$ and dim(span B) = 3 = dim(\mathcal{P}_2), we can use the stated result to conclude span $B = \mathcal{P}_2$. Therefore, B is a basis for \mathcal{P}_2 .

Here's one more potential basis for \mathcal{P}_2 , this one I just made up out of the blue.

$$B = \{x - 1, x + 1, x^2 + 2\}.$$

This set of polynomial vectors doesn't have the nice structure of the standard basis or Lagrange basis we worked with above. Regardless, let's check what

$$b(x) = \alpha_0 (x - 1) + \alpha_1 (x + 1) + \alpha_2 (x^2 + 2) = 0 \text{ for all } x$$

implies about the scalars α_0 , α_1 and α_2 . I can rewrite b(x) in terms of the standard basis

$$b(x) = (-\alpha_0 + \alpha_1 + 2\alpha_2) 1 + (\alpha_0 + \alpha_1) x + \alpha_2 x^2$$

and use the fact that $\{1, x, x^2\}$ is independent to conclude

$$b(x) = 0$$
 for all $x \iff -\alpha_0 + \alpha_1 + 2\alpha_2 = 0$, $\alpha_0 + \alpha_1 = 0$, $\alpha_2 = 0$.

Now, use your Gaussian elimination skills to see

$$-\alpha_0 + \alpha_1 + 2\alpha_2 = 0$$

$$\alpha_0 + \alpha_1 = 0 \implies \alpha_2 = 0, \ \alpha_1 = 0, \ \alpha_0 = 0.$$

$$\alpha_2 = 0$$

Therefore, $\{(x-1), (x+1), (x^2+2)\}$ is an independent set.

5. Show $\alpha_0 1 + \alpha_1 x + \cdots + \alpha_n x^n = 0$ for all x implies $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. This says $\{1, x, \ldots, x^n\}$ is a independent set in \mathcal{P}_n . Hint: Differentiate a certain number of times and then set x = 0.

6. Let x_0 , x_1 , x_2 be three distinct real numbers.

(a) Show that

$$\{(x-x_1)(x-x_2), (x-x_0)(x-x_2), (x-x_0)(x-x_1)\}$$

forms an independent set in \mathcal{P}_2 .

(b) Write an arbitrary $p(x) \in \mathcal{P}_2$ in terms of a linear combination of these three vectors. Answer:

$$p(x) = p(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + p(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + p(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

7. Determine whether or not the given sets of vectors from \mathcal{P}_2 are independent.

(a) {1, (x-0), (x-0)(x-1)} (b) {2x+1, x^2 , $(x+1)^2$ }

Answers: (a) Independent. (b) Dependent.

8. Recall the independent set of vectors from \mathcal{P}_2 I cooked up above:

$$\{x-1, x+1, x^2+2\}.$$

Write $p(x) = \alpha_0 (x - 1) + \alpha_1 (x + 1) + \alpha_2 (x^2 + 2)$ by solving for the constant scalars α_0 , α_1 and α_2 when

(a) p(x) = 1 (b) p(x) = x (c) $p(x) = x^2$ (d) $p(x) = x^2 + 3x + 4$

Answers: (a) $\alpha_0 = -1/2$, $\alpha_1 = 1/2$, $\alpha_2 = 0$. (b) $\alpha_0 = 1/2$, $\alpha_1 = 1/2$, $\alpha_2 = 0$. (c) $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_2 = 1$.

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