Vector Spaces That Are Not $\mathbb{R}^m$

Consider a vector space with the same scalars and vectors as in $\mathbb{R}^2$

scalars: $\alpha \in \mathbb{R}$ and vectors: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$

but with scalar multiplication and vector addition defined differently

$$\alpha \mathbf{x} = \begin{pmatrix} \alpha(x_1 - 1) + 1 \\ \alpha(x_2 - 1) + 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix}.$$

Let’s refer to this strange vector space as $S^2$. Of course, in order to call $S^2$ a vector space, it must be verified that structural conditions (a–0) through (d–2) listed in your previous homework are in fact true. Clearly, (a–0) is true. Moreover, vector addition is associative and commutative

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 1 + z_1 - 1 \\ x_2 + y_2 - 1 + z_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 2 \\ x_2 + y_2 + z_2 - 2 \end{pmatrix},$$

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 + z_1 - 1 \\ y_2 + z_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 1 - 1 \\ x_2 + y_2 + z_2 - 1 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_2 - 2 \\ x_2 + y_2 + z_2 - 2 \end{pmatrix},$$

$$\mathbf{y} + \mathbf{x} = \begin{pmatrix} y_1 + x_1 - 1 \\ y_2 + x_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} = \mathbf{x} + \mathbf{y}.$$

You’ll be asked to verify (a–3) and (a–4) in an exercise below. FYI:

the additive identity $\mathbf{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}$’s additive inverse $\mathbf{x}' = \begin{pmatrix} 2 - x_1 \\ 2 - x_2 \end{pmatrix}$.

Continuing with the listed items, (m–0) is clearly true. For (m–1)

$$\alpha(\beta \mathbf{x}) = \alpha \begin{pmatrix} \beta(x_1 - 1) + 1 \\ \beta(x_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} \alpha(\beta(x_1 - 1) + 1 - 1) + 1 \\ \alpha(\beta(x_2 - 1) + 1 - 1) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \beta(x_1 - 1) + 1 \\ \alpha \beta(x_2 - 1) + 1 \end{pmatrix} = (\alpha \beta) \mathbf{x},$$

and (m–2)

$$1 \mathbf{x} = \begin{pmatrix} 1(x_1 - 1) + 1 \\ 1(x_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}.$$

For (d–1)

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix} = \begin{pmatrix} \alpha(x_1 + y_1 - 1 - 1) + 1 \\ \alpha(x_2 + y_2 - 1 - 1) + 1 \end{pmatrix},$$

$$\alpha \mathbf{x} + \alpha \mathbf{y} = \begin{pmatrix} \alpha(x_1 - 1) + 1 \\ \alpha(x_2 - 1) + 1 \end{pmatrix} + \begin{pmatrix} \alpha(y_1 - 1) + 1 \\ \alpha(y_2 - 1) + 1 \end{pmatrix} = \begin{pmatrix} \alpha(x_1 - 1) + 1 + \alpha(y_1 - 1) + 1 - 1 \\ \alpha(x_2 - 1) + 1 + \alpha(y_2 - 1) + 1 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(x_1 + y_1 - 1 - 1) + 1 \\ \alpha(x_2 + y_2 - 1 - 1) + 1 \end{pmatrix},$$

and so $\alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$. You’ll be asked to verify property (d–2) in an exercise below.
1. Let \( V \) be any vector space.

(a) Suppose \( 0 \) and \( 0' \) are additive identities for \( V \). Show \( 0' = 0 \) to conclude \( V \)’s additive identity is unique.

(b) Let \( 0 \) denote \( V \)’s scalar field additive identity and let \( 0 \) denote \( V \)’s vector additive identity. For any \( x \in V \) show that \( 0x = 0 \).

(c) Suppose \( x \in V \) has additive inverses \( x' \) and \( x'' \). Show \( x'' = x' \) to conclude \( x \)’s additive inverse is unique.

(d) Let \( -1 \) denote \( V \)’s scalar field additive inverse of its multiplicative identity \( 1 \). For any \( x \in V \) show that \( -1x = x' \) where \( x' \) is \( x \)’s additive inverse.

Please justify each step by stating which properties from (a–0) through (d–2) were used.

2. Recall what I called \( S^2 \) above.

(a) Explicitly calculate the additive identity vector, \( 0 \), and the additive inverse of \( x \), \( x' \), in order to establish that \( S^2 \) satisfies vector space properties (a–3) and (a–4).

(b) For \( S^2 \) confirm that \( 0x = 0 \) and \( -1x = x' \).

(c) Show \( S^2 \) satisfies property (d–2).

I’ll add two more vector spaces and more exercises here asap. When done, it’ll be noted at the web page link.