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Final, Sp 25

1 Find all solutions

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$

Aug matrix and reduce

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 - R1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2 = 4 \quad x_1 = 2$

$x_2 = 1$

← we have a solution

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$

Aug matrix and then reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 & 6 \\ 1 & 2 & 2 & 1 & 6 \end{array}\right] \quad \begin{array}{l} R2 = R2 - R1 \\ R3 = R3 - R1 \end{array}$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right] \quad R2 \leftrightarrow R3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

This is in echelon form

$x_4 = \alpha$

$x_3 + \alpha = 2$

$x_3 = 2 - \alpha$

$x_2 + (2-\alpha) = 2$

$x_2 = \alpha$

$x_1 + \alpha + (2-\alpha) + \alpha = 4$

$x_1 = 2 - \alpha$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2-\alpha \\ \alpha \\ 2-\alpha \\ \alpha \end{pmatrix}}$$

2 Is following true

$$\begin{pmatrix} 5 \\ 6 \\ 8 \\ 3 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

or is $\begin{pmatrix} 5 \\ 6 \\ 8 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ solvable?

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Aug matrix and then reduce

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 2 & 1 & 8 \\ 1 & 0 & 3 \end{array}\right] \quad \begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 - 2R1 \\ R4 \to R4 - R1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{Solvable}$$

$\beta = 2$

$\alpha + 2 = 5$, $\alpha = 3$ $\boxed{\text{True}}$

and also as required

$$\boxed{\begin{pmatrix} 5 \\ 6 \\ 8 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}$$

b) $$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\} ?$$

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or is

$$\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \text{ solvable?}$$

Aug matrix and solve

$$\left[\begin{array}{ccc|c}1&1&0&1\\2&0&1&1\\2&1&0&1\\1&0&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&1\\0&2&-1&1\\0&1&0&1\\0&1&-1&1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&1&0&1\\0&1&0&1\\0&2&-1&1\\0&1&-1&1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&1\\0&1&0&1\\0&0&-1&-1\\0&0&-1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&1&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{array}\right] \longleftarrow \text{No Solution!}$$

$$\therefore \boxed{\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \notin \operatorname{span}\left\{\begin{pmatrix}1\\2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}}$$

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3 Determine basis for nullspace and std basis for range of $\mathcal{L}$ when

a) $\mathcal{L}(X) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

For null($\mathcal{L}$) find all solutions to $\mathcal{L}(\vec{X}) = \vec{0}$, has Aug matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} \leftarrow x_1 + 2(-2\alpha) + 3\alpha = 0 \\ \leftarrow x_2 + 2\alpha = 0 \\ \quad x_2 = -2\alpha \end{array}$$

free
$x_3 = \alpha$

$x_1 = \alpha$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Null}(\mathcal{L}) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}}$$

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For Range($\mathcal{L}$) = Span $\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$

To get Std basis

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 4 \\ 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

transpose back $\boxed{\text{Range}(\mathcal{L}) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}$

b) $\mathcal{L}(x) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

For Null space find all solns to $\mathcal{L}(\vec{x}) = \vec{0}$

Aug matrix $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = \alpha$

$x_3 = \beta$

$x_1 + 2\alpha + 3\beta = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ \beta \end{pmatrix} = \text{(over)}$$

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$$= \alpha\begin{pmatrix}-2\\1\\0\end{pmatrix} + \beta\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

$$\boxed{\mathrm{Null}(\mathcal{L}) = \mathrm{Span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}}$$

$$\mathrm{Range}(\mathcal{L}) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}\right\}$$

to get std basis transpose and reduce to canonical form

$$\begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix} \sim \begin{bmatrix}1 & 2\\0 & 0\\0 & 0\end{bmatrix}$$

$$\boxed{\mathrm{Range}(\mathcal{L}) = \mathrm{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}}$$

4 Find inverse if it exists

(a) $A = \begin{pmatrix}2 & 1\\2 & 3\end{pmatrix}$ $\det(A) = 4$ $\boxed{A^{-1} = \frac{1}{4}\begin{pmatrix}3 & -1\\-2 & 2\end{pmatrix}}$

Cramer's Rule

b) $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ $\det(A) = 0$ Not invertible

c) $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 2 \end{pmatrix}$

I'll do by elimination first

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$ $R2 = R1 - R2$, $R3 = 3R1 - 2R3$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 3 & 0 & -2 \end{array}\right]$$

6 4 4 0 0 2
6 6 3 3 0 0

$R3 = R3 - R2$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$ $R1 = R1 - 2R3$, $R2 = R2 + R3$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$ $R1 = R1 - 2R2$

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$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -4 & -4 & 6 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right] \quad R1/2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 3 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

So By elimination $\boxed{A^{-1} = \begin{pmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{pmatrix}}$

Also do This by Cramer's Rule

$$C = \begin{pmatrix} +(-2) & -(-2) & +(1) \\ -(2) & +(1) & -(-2) \\ +(3) & -(2) & +(-2) \end{pmatrix}$$

$$\det(A) = 2(-2) + 2(2) + 1(1) = \boxed{1}$$

$$A^{-1} = \frac{1}{1} C^T = \boxed{\begin{pmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{pmatrix}}$$

5 Find det of following

a) $\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{pmatrix}$ by criss-cross

$= [9+2+3] - [3+6+3]$

$= 14 - 12 = \boxed{2}$

b) $\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ ← cofactor along row 1

$= 1\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$= \{[4] - [2+1]\} - \{2-1\}$

$= 1 - 1 = \boxed{0}$

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6 Find evals & evecs.

a) $\begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$

evals: $\det\begin{pmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{pmatrix} = (\lambda+2)(\lambda-5)+12$

$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

$\lambda=1,\ \lambda=2$

Next evecs $\quad (A-\lambda I)\vec{r}=\vec{0}$

$\boxed{\lambda=1}$ $\quad \left[\begin{array}{cc|c} 5-1 & -6 & 0 \\ 2 & -2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 2 & -3 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad 2r_1 - 3\alpha = 0$

$r_2=\alpha$ $\quad r_1 = \frac{3}{2}\alpha$

WLOG take $\alpha=2$

$\boxed{r=\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

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For $\boxed{\lambda=2}$ $\quad (A-\lambda I)\vec{r}=\vec{0}$

$$\left[\begin{array}{cc|c} 5-2 & -6 & 0 \\ 2 & -2-2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 3 & -6 & 0 \\ 2 & -4 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$r_2 = \alpha$

take $\alpha = 1$ WLOG

$r_1 - 2\alpha = 0$

$r_1 = 2$

$$\boxed{\vec{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

b $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

evals $\det \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix}$

$= (2-\lambda)\left[(1-\lambda)^2 - 1\right]$ over

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$$=(2-\lambda)\left[\lambda(\lambda-2)\right]$$

$\lambda=0$ & $\lambda=2$ with mult 2,

evecs. $(A-\lambda)\vec{r}=\vec{0}$

$\lambda=0$ $\left[\begin{array}{ccc|c} 1-0 & 0 & -1 & 0 \\ 0 & 2-0 & 0 & 0 \\ -1 & 0 & 1-0 & 0 \end{array}\right]$

$$=\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_2=0$ $\quad r_3=\alpha$

$r_1=\alpha$ $\quad$ Take $\alpha=1$

$$\boxed{r=\begin{pmatrix}1\\0\\1\end{pmatrix}}$$

$(A-\lambda I)\vec{r}=\vec{0}$

$\lambda=2$ $\left[\begin{array}{ccc|c} 1-2 & 0 & -1 & 0 \\ 0 & 2-2 & 0 & 0 \\ -1 & 0 & 1-2 & 0 \end{array}\right]$

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$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_2 = \alpha \quad r_3 = \beta$

$$\begin{pmatrix} -\beta \\ \alpha \\ \beta \end{pmatrix} = \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad r_1 = -\beta$$

$\lambda = 2$ has 2 indep evecs

$$\boxed{r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

7 Consider

$$\{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

a) Show They are indep.
That is, show

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

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has Aug matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\alpha_1 = 0$ $\qquad \alpha_2 = 0$ $\qquad \alpha_3 = 0$

$\therefore$ set is independant.

b) Show that $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Span } B \subseteq \mathbb{R}^3$

$\dim(\text{span } B) = 3 \qquad \dim(\mathbb{R}^3) = 3$

$\Rightarrow \text{span}(B) = \mathbb{R}^3$

c) Use Gram Schmidt to generate orthog Basis $\{e_1\ e_2\ e_3\}$

$\ni e_1 \in \text{span}\{b_1\}$ $\quad e_2 \in \text{span}\{b_1\ b_2\}$ ... (over)

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$$\boxed{e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$e_2 = b_2 - \alpha e_1$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Find α $\ni e_2 \perp e_1$

$$0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 - 2\alpha$$

$\alpha = 1/2$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

I'll renormalize

$$\boxed{e_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

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$$e_3 = b_3 - \alpha e_1 - \beta e_2$$

Find $\alpha \ni e_3 \perp e_1$

$$0 = (e_3, e_1) = (b_3 - \alpha e_1 - \beta e_2)\cdot e_1$$

$$= b_3 \cdot e_1 + \alpha(e_1 \circ e_1)$$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} \circ \begin{pmatrix}1\\1\\0\end{pmatrix} - \alpha \begin{pmatrix}1\\1\\0\end{pmatrix} \circ \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$0 - 2\alpha \Rightarrow \boxed{\alpha = 0}$$

$$0 = (e_3 \circ e_2) = (b_3 - \alpha e_1 - \beta e_2) \circ e_2$$

$$= (b_3 \cdot e_2) - \beta(e_2 \circ e_2)$$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} \circ \begin{pmatrix}-1\\1\\0\end{pmatrix} - \beta \begin{pmatrix}-1\\1\\0\end{pmatrix} \circ \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$0 - 2\beta \quad \boxed{\beta = 0}$$

$$\boxed{e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$