Math 2318 Final Exam.   Sanders Spring 2025

This exam has **seven** problems, and all will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, **last name first**, and **student id number** on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. Find **all** solutions, if there are any, to the following.

$$\text{(a)}\quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \qquad \text{(b)}\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$$

2. Determine if the following is true or not. When true, write the given vector as a linear combination of the spanning vectors. Please show all your work.

$$\text{(a) Is } \begin{pmatrix} 5 \\ 6 \\ 8 \\ 3 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}? \quad \text{(b) Is } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}?$$

3. Determine a basis for the null space and the **standard** basis for the range space of the linear operators, $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^2$, defined as follows.

$$\text{(a)}\ \mathcal{L}(\mathbf{x}) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \qquad \text{(b)}\ \mathcal{L}(\mathbf{x}) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

4. If the given matrix is invertible, find its inverse. If not, say why. State whether you are using elimination or Cramer's rule. (Parts (a) and (b) are worth 5 points, part (c) is worth 10 points.)

$$\text{(a)}\ \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}. \qquad \text{(b)}\ \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}. \qquad \text{(c)}\ \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 2 \end{pmatrix}.$$

5. Find the determinant of each matrix.

$$\text{(a)}\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{pmatrix}. \qquad \text{(b)}\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

6. Find the eigenvalues and associated eigenvectors for the following matrices.

$$\text{(a)}\ \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}. \qquad \text{(b)}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

7. Consider the set of vectors

$$\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \equiv \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(a) (5 points) Show the vectors in this set are linearly independent.

(b) (5 points) Conclude this is a basis for $\mathbb{R}^3$.

(c) (10 points) Use Gram-Schmidt to find an <u>orthogonal</u> basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ subject to $\mathbf{e}_1 \in \text{span}\{\mathbf{b}_1\}$, $\mathbf{e}_2 \in \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{e}_3 \in \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

Hint on (b): You must show $\text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \mathbb{R}^3$ to conclude this is an independent spanning set.