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Sanders
2318 Final
Sp 26

1 Find all solutions

a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Solvable, and $x_2 = 1$

$x_1 + x_2 = 3 \Rightarrow x_1 = 2$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

b) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

Solvable and x_3 free

$x_3 = \alpha$

$x_2 + \alpha = 2$

$x_2 = 2 - \alpha$

$x_1 + (2 - \alpha) + \alpha = 3$

$x_1 = 1$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\alpha \\ \alpha \end{pmatrix}}$$

2 Find all solutions

a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

impossible.

$\boxed{\text{No Solution}}$

b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_3 = 2$
$x_2 = 0$
$x_1 + 0 + 2 = 3$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$$

(3)

3 Get basis for Null space, std basis for Range space and state dims of each

a) $\mathcal{L}(X) = \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

Null space Solve $\mathcal{L}(X) = 0$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

X_2 is free
X_3 is free

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$X_2 = \alpha$
$X_3 = \beta$
$X_1 + \alpha + \beta = 0$

$$\text{Null}(\mathcal{L}) = \text{Span} \boxed{\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

This is a basis set.

$$\boxed{\dim(\text{Null}(\mathcal{L})) = 2}$$

(4)

$$\text{Range}(\mathcal{L}) = \text{span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Span of the columns,

To get std basis consider

Need to normalize to 1

$$\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Range}(\mathcal{L}) = \text{span}\boxed{\left\{ \begin{pmatrix} 1 \\ 2/3 \\ 1/3 \end{pmatrix} \right\}}$$

and $\boxed{\dim(\text{Range}(\mathcal{L})) = 1}$

(5)

b) $\mathcal{L}(x) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solve $\mathcal{L}(x) = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = 0$

$x_1 = 0$

$\boxed{\text{Null}(\mathcal{L}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}}$

Basis set is Null set $\emptyset$. $\boxed{\dim(\text{Null}(\mathcal{L})) = 0}$

$\text{Range}(\mathcal{L}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

(Span of the columns)

std basis consider $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$\text{Range}(\mathcal{L}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ $\boxed{\dim(\text{Range}(\mathcal{L})) = 2}$

⑥ 4 Find inverse

a) $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

By Cramer's Rule

$$A^{-1} = \frac{1}{10-12}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$= \boxed{-\frac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}} = \boxed{\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}}$$

b) $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ Find inverse by elimination

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

(over)

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$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

So

$$\boxed{A^{-1} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}}$$

5 Use $\det(AB) = \det(A)\det(B)$

& $\det(U) = u_{11} u_{22} \cdots u_{nn}$

Assume S is invertible & U upper Δ.

a) Show $\det(S) \neq 0$.

Since $I = SS^{-1} \Rightarrow \det(I) = \det(SS^{-1})$

$\Rightarrow 1 = \det(S)\det(S^{-1})$ (*)

whatever $\det(S^{-1})$ is equal to

It's impossible for $\det(S) = 0$.

$\therefore \det(S) \neq 0$.

b) $A = S^{-1} U S$

$\Rightarrow \det(A) = \det(S^{-1}) \det(U) \det(S)$

from part (a) $\det(S^{-1}) \det(S) = 1$

$\det(A) = \det(U) = u_{11} u_{22} \cdots u_{nn}$

6 Find evals & evecs.

a) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \leftarrow$ This is upper Δ!

$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix}$

$= (1-\lambda)(2-\lambda)(3-\lambda)$

$\lambda = 1, \lambda = 2, \lambda = 3$

(over)

For evecs

$\lambda = 1$ $\quad (A - \lambda I)\vec{r} = \vec{0}$ $\quad \left[\begin{array}{ccc|c} 1-1 & 0 & 1 & 0 \\ 0 & 2-1 & 1 & 0 \\ 0 & 0 & 3-1 & 0 \end{array}\right]$

(9)

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

r_1 is free $r_2 = r_3 = 0$

say $r_1 = 1$

$$\boxed{\vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda = 1}$$

$\lambda = 2$ $(A - \lambda I)\vec{r} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1-2 & 0 & 1 & 0 \\ 0 & 2-2 & 1 & 0 \\ 0 & 0 & 3-2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

r_2 free say $r_2 = 1$

$r_3 = 0$

$r_1 - 0 = 0$

$$\boxed{\vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ for } \lambda = 2}$$

(10)

For $\lambda = 3$

$(A-\lambda I)\vec{r} = \vec{0}$ $\left[\begin{array}{ccc|c} 1-3 & 0 & 1 & 0 \\ 0 & 2-3 & 1 & 0 \\ 0 & 0 & 3-3 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

r_3 is free, say $r_3 = 1$

$\boxed{\vec{r} = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}}$ for $\lambda = 3$

$r_2 - 1 = 0$

$r_2 = 1$

$2r_1 - 1 = 0$

$r_1 = 1/2$

b) $A = \begin{pmatrix} -2 & 0 & 4 \\ 2 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix}$

$\det(A-\lambda I) = (2-\lambda)\det\begin{pmatrix} -2-\lambda & 4 \\ -1 & 3-\lambda \end{pmatrix}$

$= (2-\lambda)\left[(\lambda+2)(\lambda-3)+4\right] = (2-\lambda)(\lambda^2-\lambda-2)$

(over)

10b

$$= (2-\lambda)\Big((\lambda-2)(\lambda+1)\Big) = -(\lambda-2)^2(\lambda+1)$$

$\lambda = -1$ & $\lambda = 2 \leftarrow$ mult 2

simple.

$\lambda = -1$

$(A - \lambda I)\vec{r} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2+1 & 0 & 4 & 0 \\ 2 & 2+1 & 1 & 0 \\ -1 & 0 & 3+1 & 0 \end{array}\right] \sim \begin{array}{ccc} -2 & 0 & 8 \end{array} \left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 0 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda = -1 \quad \vec{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

$r_2 + 3 \cdot 1 = 0$

$r_2 = -3$

$r_1 - 4 \cdot 1 = 0$

$r_1 = 4$

r_3 is free

say $r_3 = 1$

$\lambda = 2 \quad (A - \lambda I)\vec{r} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2-2 & 0 & 4 & 0 \\ 2 & 2-2 & 1 & 0 \\ -1 & 0 & 3-2 & 0 \end{array}\right]$$

(over)

10c

$$\sim \left[\begin{array}{ccc|c} -4 & 0 & 4 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} 2 & 0 & 2 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\boxed{\lambda = 2 \quad \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

r_2 is free

Say $r_2 = 1$

$r_3 = 0$

$r_1 - 0 = 0$

(11)

$$\{b_1, b_2, b_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

a) Show set is independent.

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to show $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} \alpha_3 = 0 \\ \alpha_2 = 0 \\ \alpha_1 + 0 = 0 \end{array}$$

So must have

$\alpha_1 = \alpha_2 = \alpha_3 = 0$ $\boxed{\text{Independent}}$

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b) $S = \text{Span}\{b_1\ b_2\ b_3\}$

$S \subseteq \mathbb{R}^3$

$\dim(S) = 3$ b.c. independent

clearly $\dim(\mathbb{R}^3) = 3$

$\Rightarrow S = \mathbb{R}^3$,

i.e. $\{b_1\ b_2\ b_3\}$ independent spanning set of $\mathbb{R}^3$

c) $\boxed{e_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}}$ Gram-Schmidt

$$e_2 = \underset{b_2}{\begin{pmatrix}1\\1\\0\end{pmatrix}} - \alpha \underset{e_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}}$$

(one!)

$$0 = e_2 \cdot e_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} - \alpha \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix}$$

(13)

$$0 = 2 - \alpha 3 \Rightarrow \alpha = 2/3$$

$$e_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\left(\begin{pmatrix}3\\3\\0\end{pmatrix} - \begin{pmatrix}2\\2\\2\end{pmatrix}\right)$$

$$= \frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$ wlog take $\frac{1}{3} = 1$ ☺

take $\boxed{e_2 = \begin{pmatrix}1\\1\\-2\end{pmatrix}}$

$$e_3 = \underset{b_3}{\begin{pmatrix}0\\1\\1\end{pmatrix}} - \alpha \underset{e_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}} - \beta \underset{e_2}{\begin{pmatrix}1\\1\\-2\end{pmatrix}}$$

$$0 = e_3 \cdot e_1 \Rightarrow \alpha = \frac{\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}} = 2/3$$

$$0 = e_3 \circ e_2 \Rightarrow \beta = \frac{\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}}{\begin{pmatrix}1\\1\\-2\end{pmatrix}\circ\begin{pmatrix}1\\1\\-2\end{pmatrix}} = \frac{-1}{6}$$ (over)

(14) 4/6

$$e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{6}\left(\begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right)$$

$$= \frac{1}{6}\begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$$

WLOG take $\boxed{e_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

Our orthog basis is

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}}$$