MATH 2318 Final Exam. Sanders Spring 2026

This exam has **seven** problems, and all will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, **last name first**, and **student id number** on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. Find all solutions if there are any.

(a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$. (b) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

2. Find all solutions if there are any.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. (b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$.

3. Determine a basis for the null space and the **standard** basis for the range space, and state the dimension of each, for the linear operator $\mathcal{L}$ given by the following.

(a) $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^3$ where $\mathcal{L}(\mathbf{x}) \equiv \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(b) $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3$ where $\mathcal{L}(\mathbf{x}) \equiv \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

4. Each of the following matrices is invertible. Find the inverse. Please state for each whether you are using elimination or Cramer's rule.

(a) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

5. You may freely use these two facts about the determinants of $n \times n$ matrices.

(i) $\det(AB) = \det(A)\det(B)$.

(ii) If U is upper triangular, then $\det(U) = u_{1,1}u_{2,2}\cdots u_{n,n}$.

Suppose that S is invertible and U is upper triangular. Prove the following.

(a) $\det(S) \neq 0$ and moreover $\det(S^{-1}) = 1/\det(S)$.

(b) If $A = S^{-1}US$, then $\det(A) = u_{1,1}u_{2,2}\cdots u_{n,n}$.

6. Find the eigenvalues and associated eigenvectors for the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. (b) $\begin{pmatrix} -2 & 0 & 4 \\ 2 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix}$.

7. Consider the set of vectors

$$\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \equiv \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(a) (5 points) Show the vectors in this set are linearly independent.

(b) (5 points) Conclude this is a basis for $\mathbb{R}^3$.

(c) (10 points) Use Gram-Schmidt to find an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ subject to $\mathbf{e}_1 \in \operatorname{span}\{\mathbf{b}_1\},\ \mathbf{e}_2 \in \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\},\ \mathbf{e}_3 \in \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.