A **linear differential operator**, say $\mathcal{L}(u)$, satisfies the following property.

For any constants $c_1$ and $c_2$ and differentiable functions $u_1$ and $u_2$, we must have

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2).$$

For example

$$\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u$$

is linear since

$$\mathcal{L}(c_1u_1 + c_2u_2) = \frac{d^2(c_1u_1 + c_2u_2)}{dx^2} + a(x)\frac{d(c_1u_1 + c_2u_2)}{dx} + b(x)(c_1u_1 + c_2u_2)$$

$$= c_1\frac{d^2u_1}{dx^2} + a(x)\frac{du_1}{dx} + b(x)u_1 + c_2\left(\frac{d^2u_2}{dx^2} + a(x)\frac{du_2}{dx} + b(x)u_2\right)$$

$$= c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2).$$

For the moment, let $\mathcal{L}(u)$ denote this generic second order linear differential operator. The general solution, say $u$, to the linear ODE

$$\mathcal{L}(u) = f(x)$$

can be decomposed into two parts

$$u = u_h + u_p$$

where $\mathcal{L}(u_h) = 0$, and $\mathcal{L}(u_p) = f(x)$. $u_h$ is called the **homogeneous solution** and $u_p$ is called a **particular solution**. Clearly $u_h + u_p$ solves the inhomogeneous ODE since

$$\mathcal{L}(u) = \mathcal{L}(u_h + u_p) = \mathcal{L}(u_h) + \mathcal{L}(u_p) = 0 + f(x) = f(x).$$

Here’s a specific example. Consider

$$\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} + u \quad \text{and check that} \quad \mathcal{L}(x^2 - 2) = x^2 \quad \text{and} \quad \mathcal{L}(x\cos(x)) = -2\sin(x).$$

From this, conclude

$$\mathcal{L}(u_p) = 5x^2 + \sin(x) \quad \text{is solved by} \quad u_p = 5(x^2 - 2) - \frac{1}{2}x\cos(x).$$

Also, from your previous homework assignment,

$$\mathcal{L}(u_h) = 0 \quad \text{has as its general solution} \quad u_h = c_1\cos(x) + c_2\sin(x).$$

You should now easily see that

$$u = c_1\cos(x) + c_2\sin(x) + 5(x^2 - 2) - \frac{1}{2}x\cos(x) \quad \text{solves} \quad \frac{d^2u}{dx^2} + u = 5x^2 + \sin(x).$$
We’ll focus on techniques for finding $u_p$ (the particular solution to the inhomogeneous problem) in the remainder of these notes.

For the constant coefficient ODE, the method of undetermined coefficients, or what I call the method of guessing, is a fast and probably the easiest way to find a particular solution. Here we’ll solve

$$\frac{d^2u}{dx^2} + a \frac{du}{dx} + bu = f(x) \quad (a \text{ and } b \text{ are constants})$$

for certain right hand sides:

1. $f(x) = x^n \ (n = 0, 1, 2, \ldots)$,
2. $f(x) = e^{hx}$,
3. $f(x) = \sin(hx)$ or $\cos(hx)$.

(See https://wikipedia.org/wiki/Method_of_undetermined_coefficients for others.)

The key to employing this technique is to guess the correct form for $u_p$.

1. Given $f(x) = x^n$, try

   $$u_p(x) = x^s (\alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_0)$$

   (typically $s = 0$ but you might need $s = 1$ or $2$).

   Use $s = 0$ unless $u = 1$ solves the homogeneous ODE. Use $s = 1$ unless $u = x$ solves the homogeneous ODE. Use $s = 2$ if both $u = 1$ and $u = x$ solve the homogeneous ODE.

Here are two examples.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + u_p = x^2.$$  Try $u_p = \alpha_2 x^2 + \alpha_1 x + \alpha_0$.

Here we can take $s = 0$ since $u = 1$ does not solve the homogeneous problem. Plug in the trial and equate powers of $x$

$$x^2 = \frac{d^2u_p}{dx^2} + u_p = (2\alpha_2) + (\alpha_2 x^2 + \alpha_1 x + \alpha_0) = \alpha_2 x^2 + \alpha_1 x + \alpha_0 + 2\alpha_2$$

$$\Rightarrow \quad \alpha_2 = 1, \alpha_1 = 0, 2\alpha_2 + \alpha_0 = 0 \quad \Rightarrow \quad \alpha_2 = 1, \alpha_1 = 0, \alpha_0 = -2$$

So, $u_p = x^2 - 2$.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + \frac{du_p}{dx} = x^2.$$  Try $u_p = x (\alpha_2 x^2 + \alpha_1 x + \alpha_0)$.

Here we take $s = 1$ since $u = 1$ does solve the homogeneous problem but $u = x$ does not. Plug in and equate powers of $x$

$$x^2 = \frac{d^2u_p}{dx^2} + \frac{du_p}{dx} = (6\alpha_2 x + 2\alpha_1) + (3\alpha_2 x^2 + 2\alpha_1 x + \alpha_0)$$

$$\Rightarrow \quad 3\alpha_2 = 1, 6\alpha_2 + 2\alpha_1 = 0, 2\alpha_1 + \alpha_0 = 0 \quad \Rightarrow \quad \alpha_2 = 1/3, \alpha_1 = -1, \alpha_0 = 2.$$

So, $u_p = x^3/3 - x^2 + 2x$.  

2
(2) Given \( f(x) = e^{hx} \) \((h \neq 0)\), try
\[
u_p(x) = x^s \alpha e^{hx} \quad (\text{typically } s = 0 \text{ but you might need } s = 1 \text{ or } 2).
\]
Use \( s = 0 \) unless \( u = e^{hx} \) is a homogeneous solution. Use \( s = 1 \) unless \( u = xe^{hx} \) is a homogeneous solution. Use \( s = 2 \) otherwise.

Here are two examples.

Find a particular solution to the following.
\[
d^2u_p dx^2 + u_p = e^x.
\]
Try \( u_p = \alpha e^x \).

Here we take \( s = 0 \) since \( u = e^x \) does not solve the homogeneous problem. Plug in and equate
\[
e^x = d^2u_p dx^2 + u_p = \alpha e^x + \alpha e^x = 2\alpha e^x \quad \Rightarrow \quad \alpha = 1/2.
\]
So, \( u_p = \frac{1}{2} e^x \).

Find a particular solution to the following.
\[
d^2u_p dx^2 - u_p = e^x. \quad \text{Try } u_p = xe^{hx}.\]

Here we take \( s = 1 \) since \( u = e^x \) does solve the homogeneous problem. Plug in and equate
\[
e^x = d^2u_p dx^2 - u_p = \alpha (xe^x + 2e^x) - \alpha xe^x = 2\alpha e^x \quad \Rightarrow \quad \alpha = 1/2.
\]
So, \( u_p = \frac{1}{2} xe^x \).

(3) Given \( f(x) = \sin(hx) \) or \( \cos(hx) \) \((h \neq 0)\), try
\[
u_p(x) = x^s (\alpha \cos(hx) + \beta \sin(hx)) \quad (\text{typically } s = 0 \text{ but you might need } s = 1).
\]
Use \( s = 0 \) unless \( u = \sin(hx) \) and \( \cos(hx) \) solve the homogeneous ODE. If they do take \( s = 1 \).

Here are two examples.

Find a particular solution to the following.
\[
d^2u_p dx^2 + du_p dx = \sin(x). \quad \text{Try } u_p = \alpha \cos(x) + \beta \sin(x).
\]

Here we take \( s = 0 \) since \( u = \sin(x) \) and \( \cos(x) \) do not solve the homogeneous problem. Plug in and equate
\[
\sin(x) = d^2u_p dx^2 + du_p dx = (-\alpha \cos(x) - \beta \sin(x)) + (-\alpha \sin(x) + \beta \cos(x))
\]
\[
= (\beta - \alpha) \cos(x) + (-\beta - \alpha) \sin(x). \quad \Rightarrow \quad \alpha = -1/2, \beta = -1/2.
\]
So, \( u_p = -\frac{1}{2} \cos(x) - \frac{1}{2} \sin(x) \).
Find a particular solution to the following.

\[ \frac{d^2 u_p}{dx^2} + u_p = \sin(x). \]
Try \( u_p = x(\alpha \cos(x) + \beta \sin(x)) \).

Here we take \( s = 1 \) since \( u = \sin(x) \) and \( \cos(x) \) do solve the homogeneous problem. Plug in the trial and equate

\[
\sin(x) = \frac{d^2 u_p}{dx^2} + u_p
= (-\alpha x \cos(x) - \beta x \sin(x) - 2\alpha \sin(x) + 2\beta \cos(x)) + (\alpha x \cos(x) + \beta x \sin(x))
= -2\alpha \sin(x) + 2\beta \cos(x) \quad \Rightarrow \quad \alpha = -1/2, \beta = 0.
\]

So, \( u_p = -\frac{1}{2} x \cos(x) \).

In exercises 1 – 5, use the method of guessing to find the general solution to the given inhomogeneous, linear and constant coefficient differential equations.

Hint: To solve part (d) use linearity together with your answers from parts (a) – (c).

1. \( \frac{d^2 u}{dx^2} + 2 \frac{du}{dx} - 3u = f(x) \) where
   
   (a) \( f(x) = x^2 + 1 \)  (b) \( f(x) = \sin(x) \)
   (c) \( f(x) = e^x \)  (d) \( f(x) = x^2 + 1 + 5 e^x \)

2. \( \frac{d^2 u}{dx^2} + 4u = f(x) \) where
   
   (a) \( f(x) = 2x + 1 \)  (b) \( f(x) = \sin(x) \)
   (c) \( f(x) = \sin(2x) \)  (d) \( f(x) = 2 \sin(x) + 3 \sin(2x) \)

3. \( \frac{d^2 u}{dx^2} - 9u = f(x) \) where
   
   (a) \( f(x) = x^2 \)  (b) \( f(x) = \sin(3x) \)
   (c) \( f(x) = e^{-3x} \)  (d) \( f(x) = 9x^2 + 10 e^{-3x} \)

4. \( \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} + u = f(x) \) where
   
   (a) \( f(x) = 3x^2 + 4 \)  (b) \( f(x) = e^x \)
   (c) \( f(x) = e^{2x} \)  (d) \( f(x) = 6x^2 + 8 + 5 e^x + 6 e^{2x} \)

5. \( \frac{d^2 u}{dx^2} = f(x) \) where
   
   (a) \( f(x) = 1 \)  (b) \( f(x) = x \)
   (c) \( f(x) = x^2 \)  (d) \( f(x) = 6x^2 + 2x + 4 \)
Next I’ll expand on what you did on exercises 2(a) and (b) of your previous homework to solve the inhomogeneous problem by factorization. I’m going to assume for the moment that the ODE is constant coefficient.

Here we go. Factor
\[
\frac{d^2u}{dx^2} + a \frac{du}{dx} + bu = f(x) \quad \Rightarrow \quad \left( \frac{d}{dx} - r_2 I \right) \left( \frac{d}{dx} - r_1 I \right) u = f(x),
\]
where \( r_1 \) and \( r_2 \) are the roots of \( r^2 + ar + b = 0 \). Let \( v = \frac{du}{dx} - r_1 u \), and solve for \( v \)
\[
\frac{dv}{dx} - r_2 v = f(x) \quad \Rightarrow \quad v = c_2 e^{r_2 x} + \int_{x_0}^{x} e^{r_2 (x-z)} f(z) \, dz.
\]
Since \( f(x) \) is specified in arbitrary form here, I’ve left the inhomogeneity in terms of the definite integral with the lower limit, \( x_0 \), an arbitrary constant. Now solve for \( u \).
\[
\frac{du}{dx} - r_1 u = v \quad \Rightarrow \quad u = c_1 e^{r_1 x} + \frac{c_2}{r_2 - r_1} e^{r_2 x} + \int_{x_0}^{x} e^{r_1 (x-w)} \left( \int_{x_0}^{w} e^{r_2 (w-z)} f(z) \, dz \right) \, dw,
\]
and decomposed \( u \) into the homogeneous solution, \( u_h \), and a particular solution, \( u_p \),
\[
u = u_h + u_p, \quad u_h = c_1 e^{r_1 x} + \frac{c_2}{r_2 - r_1} e^{r_2 x}, \quad u_p = \int_{x_0}^{x} e^{r_1 (x-w)} \left( \int_{x_0}^{w} e^{r_2 (w-z)} f(z) \, dz \right) \, dw.
\]
The student who has completed Calculus 3 will recognize the double integral above represents an iterated integral over a triangular region in \( z-w \) space. The same student will know how to reverse the order of integration
\[
\int_{x_0}^{x} \int_{x_0}^{w} \cdots \, dz \, dw = \int_{x_0}^{x} \int_{z}^{x} \cdots \, dw \, dz.
\]
(Don’t worry if you haven’t done double integrals yet.) Use this in \( u_p \) above to find
\[
u_p = \int_{x_0}^{x} f(z) \left( \int_{z}^{x} e^{r_1 (x-w)} e^{r_2 (w-z)} \, dw \right) \, dz = \int_{x_0}^{x} f(z) \left( \frac{e^{r_2 (x-z)} - e^{r_1 (x-z)}}{r_2 - r_1} \right) \, dz.
\]
Now, here’s the magic. Call the bracketed term in the right integral \( u_s(x; z) \):
\[
u_s(x; z) \equiv \left( \frac{e^{r_2 (x-z)} - e^{r_1 (x-z)}}{r_2 - r_1} \right).
\]
As a function of \( x \) (think of \( z \) as a parameter) notice that \( u_s \) is the solution to the following homogeneous IVP (the initial conditions are specified at \( x = z \)).
\[
\frac{d^2u_s}{dx^2} + a \frac{du_s}{dx} + bu_s = 0 \text{ with initial conditions } u_s(z) = 0, \ u_s'(z) = 1.
\]
Therefore, a particular solution to the inhomogeneous constant coefficient problem with general right hand side \( f(x) \) is given by the very pleasing formula
\[
u_p(x) = \int_{x_0}^{x} u_s(x; z) \, f(z) \, dz.
\]
The formula just derived for a particular solution to the inhomogeneous ODE assumed the differential equation had constant coefficients. In fact however, the formula just derived is valid even when the coefficients $a$ and $b$ vary with $x$. Let me restate the result in this greater generality.

The formula

(D-1) \[ u_p(x) = \int_{x_0}^{x} u_*(x; z) f(z) \, dz, \]

where $u_*$ is found by solving the homogeneous IVP

(D-2) \[ \frac{d^2 u_*}{dx^2} + a(x) \frac{du_*}{dx} + b(x) u_* = 0 \text{ with } u_*(z) = 0, \ u'_*(z) = 1, \]

gives a particular solution to the inhomogeneous ODE

(D-3) \[ \frac{d^2 u}{dx^2} + a(x) \frac{du}{dx} + b(x) u = f(x). \]

I’ll verify this formula is valid for the variable coefficient problem immediately following the next group of exercises.

This is the second order ODE variant of what has come to be known as Duhamel’s principle; Jean-Marie Duhamel: https://wikipedia.org/wiki/Duhamel’s_principle. Loosely speaking the principle says ”the solution of the linear and homogeneous IVP can be used to generate the solution of the inhomogeneous problem.” This carries over to a variety of well known PDEs, integral equations and linear ODEs of any order.

Here’s a constant coefficient example. It’s easy to compute (you do it) that the solution to the homogeneous IVP

\[ \frac{d^2 u_*}{dx^2} + u_* = 0, \ u_*(z) = 0, \ u'_*(z) = 1, \text{ is } u_*(x) = \sin(x - z). \]

Therefore, according to (D-1), (D-2), (D-3)

\[ u_p(x) = \int_{0}^{x} \sin(x - z) \sin(z) \, dz \text{ solves } \frac{d^2 u_p}{dx^2} + u_p = \sin(x). \]

(I took $x_0 = 0$ for convenience.) To evaluate the integral, use

\[ \sin(a) \sin(b) = \frac{1}{2} (\cos(a - b) - \cos(a + b)), \]

and compute

\[ \int_{0}^{x} \sin(x - z) \sin(z) \, dz = \frac{1}{2} \int_{0}^{x} (\cos(x - 2z) - \cos(x)) \, dz = \frac{1}{2} \sin(x) - \frac{1}{2} x \cos(x). \]

Look back at the top of page 4 where a particular solution is found by the method of guessing.
Here’s a variable coefficient example.

$$\frac{d^2 u_*}{dx^2} + \frac{1}{x} \frac{du_*}{dx} - \frac{1}{x^2} u_* = 0, \ u_*(z) = 0, \ u'_*(z) = 1, \ \Rightarrow \ u_*(x) = \frac{1}{2} \left( x - \frac{z^2}{x} \right).$$

Therefore,

$$u_p(x) = \int_1^x \frac{1}{2} \left( x - \frac{z^2}{x} \right) \sqrt{z} \, dz \text{ solves } \frac{d^2 u_p}{dx^2} + \frac{1}{x} \frac{du_p}{dx} - \frac{1}{x^2} u_p = \sqrt{x}.$$  

(I took $x_0 = 1$ to avoid division by zero.) Evaluate the integral to get

$$u_p(x) = \frac{4}{21} x^{5/2} - \left( \frac{x}{3} - \frac{1}{7x} \right).$$

6. Use Duhamel’s formula, (D-1), (D-2), (D-3), to find a particular solution to the following inhomogeneous problems. (Take your lower limit $x_0 = 0$.)

(a) $\frac{d^2 u}{dx^2} + u = \cos(x)$  \hspace{1cm}  (c) $\frac{d^2 u}{dx^2} = x^2$

(b) $\frac{d^2 u}{dx^2} - u = e^{-x}$  \hspace{1cm}  (d) $\frac{d^2 u}{dx^2} - \frac{du}{dx} = e^x$

Answers:  
(a) $u(x) = \frac{1}{2} x \sin(x).$  \hspace{1cm}  (b) $u(x) = \frac{1}{2} \sinh(x) - \frac{1}{2} x e^{-x}.$  \hspace{1cm}  (c) $u(x) = \frac{1}{12} x^4.$

(d) $u(x) = xe^x - e^x + 1.$

7. Use Duhamel to find a particular solution and then determine the general solution for

$$\frac{d^2 u}{dx^2} - 2 \frac{du}{dx} + 2 u = \sin(x).$$

Your integral should look like $u(x) = \int_{x_0}^x e^{x-z} \sin(x-z) \sin(z) \, dz = \ldots$ The integration here may take some effort.

8. Do the same as exercise 6. (Take your lower limit $x_0 = 1$ this time however.)

(a) $\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = x$  \hspace{1cm}  (b) $\frac{d^2 u}{dx^2} - \frac{1}{x} \frac{du}{dx} = x$

Answers:  
(a) $u(x) = -\frac{1}{3} \log(x) + \frac{1}{3} (x^3 - 1).$  \hspace{1cm}  (b) $u(x) = \frac{1}{3} x^3 - \frac{1}{2} x^2 + \frac{1}{6}.$

Here’s a terse verification of the Duhamel formula for the variable coefficient problem.

The Leibniz integral rule states that

$$\frac{d}{dx} \int_{x_0}^x f(x, y) \, dy = f(x, x) + \int_{x_0}^x \frac{\partial}{\partial x} f(x, y) \, dy.$$  

It’s not hard to derive this under reasonable hypotheses.

Apply the Leibniz rule to formula (D-1)

$$\frac{d}{dx} u_p(x) = \frac{d}{dx} \int_{x_0}^x u_*(x; z) f(z) \, dz = u_*(x; x) f(x) + \int_{x_0}^x \frac{\partial}{\partial x} u_*(x; z) f(z) \, dz.$$
Now, the initial condition \( u_*(z) = 0 \) in (D-2) is shorthand for \( u_*(x; z)|_{x=z} = 0 \). Since this is true for any \( z \), we must have \( u_*(x; x) = 0 \). Therefore
\[
\frac{d}{dx} u_p(x) = \int_{x_0}^{x} \frac{\partial}{\partial x} u_*(x; z) f(z) \, dz.
\]

Apply the Leibniz rule to this again
\[
\frac{d^2}{dx^2} u_p(x) = \frac{\partial}{\partial x} u_*(x; x) f(x) + \int_{x_0}^{x} \frac{\partial^2}{\partial x^2} u_*(x; z) f(z) \, dz.
\]
The initial condition \( u_*(z) = 1 \) in (D-2) is shorthand for \( (d/dx) u_*(x; z)|_{x=z} = 1 \). Since this is true for any \( z \), conclude this time that \( \left( \frac{\partial}{\partial x} \right) u_*(x; x) = 1 \). Therefore
\[
\frac{d^2}{dx^2} u_p(x) = f(x) + \int_{x_0}^{x} \frac{\partial^2}{\partial x^2} u_*(x; z) f(z) \, dz.
\]
Plug these into the left hand side of (D-3) to get
\[
\frac{d^2}{dx^2} u_p(x) + a(x) \frac{du_p}{dx} + b(x) u_p = f(x) + 0 = f(x).
\]

**How to choose the lower limit \( x_0 \)**

From the previous paragraph we have shown
\[
\begin{align*}
u_p(x) &= \int_{x_0}^{x} u_*(x; z) f(z) \, dz, \\
u_p'(x) &= \int_{x_0}^{x} \frac{\partial}{\partial x} u_*(x; z) f(z) \, dz.
\end{align*}
\]
As you all know, \( \int_{x_0}^{x} \cdots = 0 \). Therefore, our Duhamel particular solution will automatically satisfy
\[
u_p(x_0) = 0, \quad u_p'(x_0) = 0.
\]

Here’s an example to show how the choice of lower limit can be exploited to solve the IVP.

Solve the inhomogeneous IVP
\[
\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u = \sqrt{x}, \quad \text{with initial conditions} \quad u(1) = 2, \ u'(1) = 3.
\]
In a previous example, we found
\[
\begin{align*}
u_p(x) &= \int_{1}^{x} \left( \frac{1}{2} \left( x - \frac{z^2}{x} \right) \right) \sqrt{z} \, dz = \frac{4}{21} x^{5/2} - \left( \frac{x}{3} - \frac{1}{7x} \right).
\end{align*}
\]
Check that as stated above, we must have \( u_p(1) = u_p'(1) = 0 \). This C-E equation has characteristic roots \( r = \pm 1 \) which gives homogeneous solution \( u_h(x) = c_1 x + c_2 x^{-1} \), so
the general solution is

\[ u(x) = c_1 x + c_2 x^{-1} + \left( \frac{4}{21} x^{5/2} - \left( \frac{x}{3} - \frac{1}{7x} \right) \right). \]

Now let’s use the initial conditions to determine \( c_1 \) and \( c_2 \).

\[ 2 = u(1) = c_1 + c_2 + 0 \]
\[ 3 = u'(1) = c_1 - c_2 + 0 \]
\[ \Rightarrow c_1 = \frac{5}{2}, c_2 = -\frac{1}{2}. \]

So, the given inhomogeneous IVP’s solution is

\[ u(x) = \frac{5}{2} x - \frac{1}{2} x^{-1} + \frac{4}{21} x^{5/2} - \left( \frac{x}{3} - \frac{1}{7x} \right). \]

9. Use your Duhamel results from exercise 6 to solve the following inhomogeneous IVPs.

(a) \( \frac{d^2 u}{dx^2} + u = \cos(x), \ u(0) = 1, \ u'(0) = 2. \)

(b) \( \frac{d^2 u}{dx^2} - u = e^{-x}, \ u(0) = 1, \ u'(0) = 2. \)

10. Use your Duhamel results from exercise 8 to solve the following inhomogeneous IVPs.

(a) \( \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = x, \ u(1) = 1, \ u'(1) = 2. \)

(b) \( \frac{d^2 u}{dx^2} - \frac{1}{x} \frac{du}{dx} = x, \ u(1) = 1, \ u'(1) = 2. \)