A \textit{vector space}, or sometimes called a \textit{linear space}, is an abstract system composed of a set of objects called vectors, an associated field of scalars, together with the operations of vector addition and scalar multiplication. Let $V$ denote the set of vectors and $\mathcal{F}$ denote the field of scalars. Here I’ll use bold lowercase Roman letters to signify vectors, i.e. $\mathbf{x} \in V$, and lowercase Greek letters to signify scalars, i.e. $\alpha \in \mathcal{F}$.

I’m going to list out now what properties vector addition and scalar multiplication are required to satisfy on a given vector space.

(a–0) For every $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\mathbf{x} + \mathbf{y} \in V$.

(a–1) For every $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z} \in V$ we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

(a–2) For every $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(a–3) There is a vector $\mathbf{0} \in V$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in V$.

(a–4) For every $\mathbf{x} \in V$ there is a vector $\mathbf{x}' \in V$ such that $\mathbf{x} + \mathbf{x}' = \mathbf{0}$.

(m–0) For every $\alpha \in \mathcal{F}$ and $\mathbf{x} \in V$ we have $\alpha \mathbf{x} \in V$.

(m–1) For every $\alpha$ and $\beta \in \mathcal{F}$ we have $\alpha (\beta \mathbf{x}) = (\alpha \beta) \mathbf{x}$ for every $\mathbf{x} \in V$.

(m–2) If $1 \in \mathcal{F}$ is the scalar field’s multiplicative identity then $1 \mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in V$.

(d–1) For every $\alpha \in \mathcal{F}$ and $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\alpha (\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$.

(d–2) $(\alpha + \beta) \mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$

Let me say some words about these items. (a–0) says $V$ is \textit{closed} under vector addition. (m–0) says $V$ is also closed under scalar multiplication. (a–1) and (a–2) say vector addition must be associative and commutative. (a–3) says $V$ must contain the additive identity. (a–4) says every vector in $V$ has its additive inverse in $V$. (d–1) and (d–2) are the scalar–vector distributive properties.

When $\mathcal{F} = \mathbb{R}$ one often calls the vector space a \textit{real} vector space, and it’s often called a \textit{complex} vector space when $\mathcal{F} = \mathbb{C}$.

In this course we will study only one particular type of vector space. The vectors themselves are column matrices

$$
\mathbf{x} = \begin{pmatrix}
x_1 \\
\vdots \\
x_m
\end{pmatrix},
$$
and the scalar field will be either the real or complex numbers. For now, let’s assume all numbers are real. Vector addition and scalar multiplication is defined exactly as done for matrices on your previous homework,

\[ x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \ \alpha \in \mathbb{R} \Rightarrow \ x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{pmatrix} \text{ and } \ \alpha x = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_m \end{pmatrix}. \]

I will use the symbol \( \mathbb{R}^m \) to denote the vector space whose vectors are composed of all such real \( m \times 1 \) matrices. Please check on your own that \( \mathbb{R}^m \) satisfies all requirements (a–0) thru (d–2) listed above.

Suppose \( \mathcal{V} \) is a vector space, i.e. it is composed of a given set of vectors \( \mathcal{V} \) with associated scalar field \( \mathcal{F} \), and its notion of vector addition and scalar multiplication satisfies requirements (a–0) thru (d–2). A subspace \( \mathcal{S} \) of \( \mathcal{V} \), denoted by \( \mathcal{S} \subseteq \mathcal{V} \), shares the same scalar field \( \mathcal{F} \) with the parent space \( \mathcal{V} \), and inherits the notion of vector addition and scalar multiplication from the parent, but its vectors are composed of a subset \( \mathcal{S} \subseteq \mathcal{V} \) of \( \mathcal{V} \)’s vectors. However, a subspace is not just a subset of vectors from a vector space, it is more. If we say \( \mathcal{S} \) is a subspace of \( \mathcal{V} \), then it must also be a vector space on its own.

Here are three examples to help clarify what a subspace is. Consider a subset of vectors from the vector space \( \mathbb{R}^2 \)

\[ S = \{ x \in \mathbb{R}^2 : x_1 = 1 \}. \]

Does this set define a subspace of \( \mathbb{R}^2 \)? The answer is no. \( S \) is neither closed under vector addition nor scalar multiplication. For example

\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \ y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S, \text{ but } \ x + y = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \notin S,
\]

and also

\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \ \alpha = 0 \in \mathbb{R} \text{ but } \alpha x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S.
\]

Here’s a second example. Consider

\[ S = \{ x \in \mathbb{R}^2 : x_1 > 0 \}. \]

This set is closed under vector addition since for arbitrary \( x \in S \) and \( y \in S \)

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S \Rightarrow x_1 > 0, \ y_1 > 0 \Rightarrow \ x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in S,
\]

since \( x_1 + y_1 > 0 \). But it’s not closed under scalar multiplication since for example

\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \ \alpha = 0 \in \mathbb{R} \text{ but } \alpha x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S.
\]

Therefore, \( S = \{ x \in \mathbb{R}^2 : x_1 > 0 \} \) does not define as subspace of \( \mathbb{R}^2 \).
Here’s a third example. Consider

\[ S = \{ \mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 0 \}. \]

This set is closed under vector addition since

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S \quad \Rightarrow \quad x_1 + x_2 = 0, \quad y_1 + y_2 = 0 \]

\[ \Rightarrow \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in S. \]

Note \( \mathbf{x} + \mathbf{y} \in S \) because \((x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0\). Moreover

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S \quad \Rightarrow \quad x_1 + x_2 = 0 \quad \Rightarrow \quad \alpha \in \mathbb{R}, \quad \alpha \mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in S, \]

since \((\alpha x_1) + (\alpha x_2) = \alpha (x_1 + x_2) = \alpha 0 = 0\). So, \( S \) is also closed under scalar multiplication. As you’ll see in a moment, this is enough to conclude \( S = \{ \mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 0 \} \) does define a subspace of \( \mathbb{R}^2 \).

On their own, the nine requirements I listed above, (a–0) thru (d–2), are independent of each other. However, for a subset of vectors which inherits its structure (addition, etc.) from a parent vector space, some requirements become redundant. Clearly, all requirements except (a–0), (a–3), (a–4) and (m–0) are automatically satisfied for such a subsystem. In fact, as you will show in an exercise, if \( S \) is a nonempty subset of vectors from a parent vector space and \( S \) is closed under the parent’s addition and scalar multiplication, i.e. (a–0) and (m–0) are true, then (a–3) and (a–4) must also be true.

This allows us to state the following.

If \( S \) is a nonempty subset of vectors from a parent vector space and \( S \) is closed wrt the parent’s vector addition and scalar multiplication, then \( S \) defines a subspace of the parent space.

1. Suppose \( S \) is a nonempty subset of vectors from a (real) vector space. Also, suppose \( S \) is closed wrt the parent’s vector addition and scalar multiplication.
   (a) Show the additive identity \( \mathbf{0} \) is in \( S \).
   (b) Show that for any \( \mathbf{x} \in S \) there is a \( \mathbf{x}' \in S \) such that \( \mathbf{x} + \mathbf{x}' = \mathbf{0} \).

   Hint: (a) \((1 + 0) \mathbf{x} = \mathbf{x}\). (b) \( \mathbf{x}' \equiv -1 \mathbf{x} \in S \).

2. Determine whether or not the following sets define a subspace of \( \mathbb{R}^2 \):
   (a) \( \{ \mathbf{x} \in \mathbb{R}^2 : x_1 = 0 \} \)  \( \quad \quad \)  \( \quad \quad \)  \( \text{(c) } \{ \mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \geq 0 \} \)
   (b) \( \{ \mathbf{x} \in \mathbb{R}^2 : x_1 - x_2 = 0 \} \)  \( \quad \quad \)  \( \quad \quad \)  \( \text{(d) } \{ \mathbf{x} \in \mathbb{R}^2 : x_1 + 2x_2 = 0 \} \)

   Either prove the set is closed under both vector addition and scalar multiplication or give an example to show one is not.
Consider a set of \( n \) vectors \( \{x_1, \ldots, x_n\} \). This set is called a \textit{dependent set} if there are \( n \) scalars, \( \alpha_1, \ldots, \alpha_n \), which are \underline{not} all zero such that

\[
\alpha_1 x_1 + \cdots + \alpha_n x_n = 0.
\]

A set of vectors that is not dependent is called an \textit{independent set}.

Given that the vectors in the set above come from the vector space \( \mathbb{R}^m \), we can use matrix elimination to determine whether the set is independent or not. The problem can be recast as follows.

\[
\alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \iff \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Make sure you work this out on your own. Check that the \( j \)th column of the \( m \times n \) matrix on the right is the column vector \( x_j \in \mathbb{R}^m \). The zero matrix on the right has size \( m \times 1 \). If the only solution to this linear system is \( \alpha_1 = \cdots = \alpha_n = 0 \), then the set is \underline{independent}. If the system has a \textit{nontrivial solution} however, the set is \underline{dependent}.

Consider the following four vectors from \( \mathbb{R}^3 \).

\[
x_1 \equiv \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad x_2 \equiv \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad x_3 \equiv \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \quad x_4 \equiv \begin{pmatrix} 7 \\ 8 \\ 3 \end{pmatrix}.
\]

I’m going to use these in the next two examples.

Is the set \( \{x_1, x_2, x_3\} \) an independent set? The augmented matrix to consider is

\[
\begin{pmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Back substitution tells us \( \alpha_3 = \alpha, \ \alpha_2 = -2\alpha \) and \( \alpha_1 = -4(-2\alpha) - 7(\alpha) = \alpha \) for any real number \( \alpha \). WLOG take \( \alpha = 1 \) to see \( 1x_1 - 2x_2 + 1x_3 = 0 \), and conclude \( \{x_1, x_2, x_3\} \) is \underline{not} an independent set of vectors.

Is the set \( \{x_1, x_2, x_4\} \) an independent set? The augmented matrix to consider here is

\[
\begin{pmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -18 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

This time back substitution tells us \( \alpha_3 = 0, \ \alpha_2 = 0 \) and \( \alpha_1 = 0 \). Therefore

\[
\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4 = 0 \ \Rightarrow \ \alpha_1 = \alpha_2 = \alpha_3 = 0,
\]

and we conclude \( \{x_1, x_2, x_4\} \) \underline{is} an independent set of vectors.
3. Prove the following. A set of vectors \( \{x_1, \ldots, x_n\} \) (assume \( n \geq 2 \)) is dependent if and only if at least one its vectors can be written as a linear combination of the others.

Hint: Consider \( x_{i_*} = \sum_{k \neq i_*} \alpha_k x_k \) for some index \( 1 \leq i_* \leq n \).

Consider the following vectors from \( \mathbb{R}^4 \).

\[
\begin{align*}
x_1 &= \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \\
x_2 &= \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \\
x_3 &= \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}, \\
x_4 &= \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}.
\end{align*}
\]

4. Is \( \{x_1, x_2, x_3\} \) an independent set of vectors?

5. Is \( \{x_1, x_2, x_4\} \) an independent set of vectors?

6. Is \( \{x_1, x_3, x_4\} \) an independent set of vectors?

Consider a finite set of vectors from \( \mathbb{R}^m \), \( \{x_1, \ldots, x_n\} \). The span of this set is the subspace of \( \mathbb{R}^m \) defined by

\[
\text{span}\{x_1, \ldots, x_n\} = \{\sum_{k=1}^{n} \alpha_k x_k : \text{each } \alpha_k \in \mathbb{R}\}.
\]

That is,

\[
y \in \text{span}\{x_1, \ldots, x_n\} \iff y = \alpha_1 x_1 + \cdots + \alpha_n x_n
\]

for some set of real numbers \( \alpha_1, \ldots, \alpha_n \). In other words, a vector is in \( \text{span}\{x_1, \ldots, x_n\} \) when it can be written as a linear combination of the specified vectors \( x_1, \ldots, x_n \).

Clearly \( \text{span}\{x_1, \ldots, x_n\} \) is closed under vector addition and scalar multiplication and is therefore a subspace of \( \mathbb{R}^m \) regardless of what the set \( \{x_1, \ldots, x_n\} \) is.

If \( \{x_1, \ldots, x_n\} \) is an independent set of vectors and \( y \in \text{span}\{x_1, \ldots, x_n\} \) then the decomposition \( y = \alpha_1 x_1 + \cdots + \alpha_n x_n \) is unique. Let me show you why. Suppose there are two ways to decompose \( y \), say

\[
y = \alpha_1 x_1 + \cdots + \alpha_n x_n \quad \text{and} \quad y = \beta_1 x_1 + \cdots + \beta_n x_n
\]

\[
\Rightarrow 0 = (\alpha_1 - \beta_1) x_1 + \cdots + (\alpha_n - \beta_n) x_n
\]

\[
\Rightarrow (\alpha_1 - \beta_1) = \cdots = (\alpha_n - \beta_n) = 0.
\]

This last step follows from the fact that \( \{x_1, \ldots, x_n\} \) is an independent set. So, since we have \( \alpha_k = \beta_k \) for each \( k = 1, \ldots, n \), the two decompositions above are in fact identical.
It’s not hard to show the following. If \( \{x_1, \ldots, x_n\} \) is a dependent set of vectors and \( y \in \text{span}\{x_1, \ldots, x_n\} \) then the decomposition \( y = \alpha_1 x_1 + \cdots + \alpha_n x_n \) is not unique. You are asked to show this in exercise 7 below.

Now, how do we compute whether or not a given vector is in a span? We’ll use elimination of course. Consider the subspace \( S \equiv \text{span}\{x_1, x_2, x_3\} \subseteq \mathbb{R}^4 \) where

\[
x_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}.
\]

Is \( y \equiv \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} \) \( \in \text{span}\{x_1, x_2, x_3\} \)? The linear system we have to solve is

\[
\begin{pmatrix} 1 & 7 & -2 \\ 4 & 10 & 1 \\ 2 & -4 & 5 \\ -3 & -1 & -4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 7 & -2 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 0 \\ 0 \end{pmatrix},
\]

and we eliminate the augmented matrix to obtain

\[
\sim \begin{pmatrix} 1 & 7 & -2 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Now, use back substitution. See that \( \alpha_3 \) is a free variable, so let \( \alpha_3 = \alpha \) where \( \alpha \) is any real number. Then, \( \alpha_2 = \frac{1}{2}(1 + \alpha) \) and \( \alpha_1 = \frac{1}{2}(11 - 3\alpha) \). So we get

\[
y = \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} = \frac{1}{2}(11 - 3\alpha) x_1 + \frac{1}{2}(1 + \alpha) x_2 + \alpha x_3 \in S.
\]

Therefore we see \( y \in S \). Moreover, since the decomposition is not unique, i.e. \( \alpha \) here can be any real number, we also conclude the set of vectors \( \{x_1, x_2, x_3\} \) is not independent. (Look back at exercise 5 above.)

Let me change \( y \) by a little bit and ask the same question.

Is \( y \equiv \begin{pmatrix} 9 \\ 27 \\ 9 \\ -16 \end{pmatrix} \in \text{span}\{x_1, x_2, x_3\} \)? The augmented matrix to consider here is

\[
\begin{pmatrix} 1 & 7 & -2 & 9 \\ 4 & 10 & 1 & 27 \\ 2 & -4 & 5 & 9 \\ -3 & -1 & -4 & -16 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & -2 & 9 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 11/10 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & -2 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

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However, the third row in the right above says $0\alpha_1 + 0\alpha_2 + 0\alpha_3 = 1/10$, and this is impossible. Therefore, this time $y \not\in \text{span}\{x_1, x_2, x_3\}$.

7. Suppose $\{x_1, \ldots, x_n\}$ is a dependent set and $y \in \text{span}\{x_1, \ldots, x_n\}$. Prove there are an infinite number decompositions such that $y = \alpha_1 x_1 + \cdots + \alpha_n x_n$.

Hint. Since $\{x_1, \ldots, x_n\}$ is a dependent set, there are numbers $\beta_1, \ldots, \beta_n$ which are not all zero such that $\beta_1 x_1 + \cdots + \beta_n x_n = 0$.

8. Let $\{x_1, x_2, x_3\}$ come from exercise 4 above. Determine if the given vector $y$ is in $\text{span}\{x_1, x_2, x_3\}$. If it is, write down and check the decomposition $y = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$.

(a) $y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $y = \begin{pmatrix} 6 \\ 15 \\ 3 \\ -18 \end{pmatrix}$

A basis for a vector space is a linearly independent spanning set. That is,

$\{b_1, \ldots, b_n\}$ is a basis for a vector space $\mathcal{V}$ if:

1. $\{b_1, \ldots, b_n\}$ is an independent set.

2. $\mathcal{V} = \text{span}\{b_1, \ldots, b_n\}$.

The dimension of a vector space is the number of basis vectors needed to span it. It’s not obvious, but this number is independent of any particular spanning basis.

Clearly,

$\mathbb{R}^2 = \text{span}\{e_1, e_2\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

and so $\mathbb{R}^2$ is two dimensional (Duh). Not as obvious, here’s another basis for $\mathbb{R}^2$

$\mathbb{R}^2 = \text{span}\{b_1, b_2\}$, where $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The basis $\{e_1, e_2\}$ is called the standard basis for $\mathbb{R}^2$. The standard basis for $\mathbb{R}^m$ is

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{m-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad e_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$.

One might think that the standard basis for $\mathbb{R}^m$ is the most useful of all of its bases. But it really depends on the application. Later in this course we will consider others.
Let me close out this assignment by showing you, by example, how to convert a given basis for a subspace of \( \mathbb{R}^m \) to its standard basis.

Recall from exercise 4 you showed

\[
x_1 = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix},
\]

is an independent set. Therefore \( \{x_1, x_2, x_3\} \) is a basis for \( S \equiv \text{span}\{x_1, x_2, x_3\} \). To determine \( S \)'s standard basis, write out an augmented matrix using these three column vectors as rows

\[
\begin{pmatrix} 1 & 4 & 2 & -3 \\ 7 & 10 & -4 & -1 \\ -2 & 1 & 5 & -14 \end{pmatrix}.
\]

Notice there’s no vertical bar (\( | \)) here. Now, row reduce to row echelon form

\[
\begin{pmatrix} 1 & 4 & 2 & -3 \\ 0 & 18 & 18 & -20 \\ 0 & 9 & 9 & -20 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 2 & -3 \\ 0 & 1 & 1 & -20/18 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Notice on the right I’ve scaled all pivots to one. Finally, starting from the right most pivot, use backward elimination to get

\[
\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

This is called the row canonical form or sometimes the reduced row echelon form for the augmented matrix. See [https://en.wikipedia.org/wiki/Row_echelon_form](https://en.wikipedia.org/wiki/Row_echelon_form). The standard basis for the subspace \( S \equiv \text{span}\{x_1, x_2, x_3\} \) can now be read off as follows

\[
S = \text{span}\{e_1, e_2, e_3\}, \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

BTW. I checked my calculation by observing

\[
x_1 = e_1 + 4e_2 - 3e_3,
\]

\[
x_2 = 7e_1 + 10e_2 - e_3,
\]

\[
x_3 = -2e_1 + e_2 - 14e_3.
\]

9. Find the standard basis for \( \text{span}\{x_1, x_3, x_4\} \) from exercise 6.

10. The set \( \{x_1, x_2, x_4\} \) from exercise 5 is not independent. However, it’s still possible to determine the standard basis for \( \text{span}\{x_1, x_2, x_4\} \) as just done. You’ll get a zero row when
eliminating to row canonical form. Disregard the zero row when you read off your basis. What is the dimension of \( \text{span}\{x_1, x_2, x_4\} \)? Answer: two.