The cofactor expansion, sometimes called the Laplace expansion named after Pierre-Simon Laplace, is one way to view and compute the determinant. Here’s how it works. Let $A$ be an $m \times m$ matrix and from it define its $i,j$th minor matrix, say $M_{i,j}$, as the $(m-1) \times (m-1)$ matrix formed by removing $A$’s $i$th row and $j$th column. For example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow M_{1,2} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}, \quad M_{2,3} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}. $$

Now, for any row, say $i_*$, the cofactor expansion says

$$\det(A) = \sum_{j=1}^{m} a_{i_*,j} (-1)^{i_*+j} \det(M_{i_*,j}).$$

FYI: The number $c_{i,j} \equiv (-1)^{i+j} \det(M_{i,j})$ is called the $i,j$th cofactor of $A$.

Using the first row in the example above, we find

$$\det \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right) = 1 (-1)^{1+1} \det \left( \begin{array}{cc} 5 & 6 \\ 8 & 9 \end{array} \right) + 2 (-1)^{1+2} \det \left( \begin{array}{cc} 4 & 6 \\ 7 & 9 \end{array} \right) + 3 (-1)^{1+3} \det \left( \begin{array}{cc} 4 & 5 \\ 7 & 8 \end{array} \right)$$

$$= +1 (45 - 48) - 2 (36 - 42) + 3 (32 - 35) = 0.$$ 

Since $\det(A) = \det(A^T)$, we can also use a cofactor expansion along any column $j_*$

$$\det(A) = \sum_{i=1}^{m} a_{i,j_*} (-1)^{i+j_*} \det(M_{i,j_*}).$$

Using the second column in the example above, we find

$$\det \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right) = 2 (-1)^{1+2} \det \left( \begin{array}{cc} 4 & 6 \\ 7 & 9 \end{array} \right) + 5 (-1)^{2+2} \det \left( \begin{array}{cc} 1 & 3 \\ 7 & 9 \end{array} \right) + 8 (-1)^{3+2} \det \left( \begin{array}{cc} 1 & 3 \\ 4 & 6 \end{array} \right)$$

$$= -2 (36 - 42) + 5 (9 - 21) - 8 (6 - 12) = 0.$$ 

When compared to Leibniz’s formula for the determinant, these cofactor expansions offer an easy to remember formula you can use to compute $\det(A)$ when $A$ is $4 \times 4$ or larger. To reduce your work, pick the row or column that has the greatest number of zeros in it.
Here’s a $4 \times 4$ example. Let
\[
A = \begin{pmatrix}
1 & 1 & 0 & 2 \\
0 & 1 & 1 & 2 \\
1 & 1 & 2 & 3 \\
2 & 1 & 0 & 1 \\
\end{pmatrix},
\]
and cofactor along the third column
\[
\det(A) = 0 (-1)^{1+3} \det \begin{pmatrix}
0 & 1 & 2 \\
1 & 1 & 3 \\
2 & 1 & 1 \\
\end{pmatrix}
+ 1 (-1)^{2+3} \det \begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & 3 \\
2 & 1 & 1 \\
\end{pmatrix}
+ 2 (-1)^{3+3} \det \begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 2 \\
2 & 1 & 1 \\
\end{pmatrix}
+ 0 (-1)^{4+3} \det \begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 2 \\
1 & 1 & 3 \\
\end{pmatrix}.
\]
There’s no need to compute the first and fourth $3 \times 3$ determinants on the right since they are both multiplied by zero. So
\[
\det(A) = 1 (-1)^{2+3} \det \begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & 3 \\
2 & 1 & 1 \\
\end{pmatrix}
+ 2 (-1)^{3+3} \det \begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 2 \\
2 & 1 & 1 \\
\end{pmatrix}
= -1 (1 + 6 + 2 - 4 - 3 - 1) + 2 (1 + 4 - 4 - 2) = -3.
\]
I’m now going to derive the cofactor determinant formula for the special case when we cofactor along the rightmost column. The more general column cofactor formulae are deduced from this by interchanging columns. Finally, the row cofactor formulae are deduced from the column cofactor formulae by means of the transpose.

Consider an $m \times m$ matrix $A$. I will show
\[
\sum_{i=1}^{m} a_{i,m} (-1)^{i+m} \det(M_{i,m}) = \det(A).
\]
To this end, recall $M_{i,m}$ is constructed by removing the $i$th row and $m$th column from $A$
\[
M_{i,m} = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,m-1} \\
\vdots & & \vdots \\
a_{i-1,1} & \cdots & a_{i-1,m-1} \\
a_{i+1,1} & \cdots & a_{i+1,m-1} \\
\vdots & & \vdots \\
a_{m,1} & \cdots & a_{m,m-1} \\
\end{pmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}
\]
\[
\Rightarrow \det(M_{i,m}) = \sum_{j \in \mathcal{P}_{m-1}} \text{sgn}(j) a_{1,j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_i} \cdots a_{m,j_{m-1}}.
\]
Use this to write

2
\[
\sum_{i=1}^{m} a_{i,m} (-1)^{i+m} \det(M_{i,m}) \\
= \sum_{i=1}^{m} \sum_{j \in \mathcal{P}_{m-1}} (-1)^{i+m} \text{sgn}(j) a_{1,j_1} \cdots a_{i-1,j_{i-1}} a_{i,m} a_{i+1,j_i} \cdots a_{m,j_{m-1}}.
\]

Now, append \( m \) to the right side of permutation \( j \) and check that

\[
j \in \mathcal{P}_{m-1}, \quad j = (j_1, \ldots, j_{i-1}, j_i, \ldots, j_{m-1}).
\]

\[
\tilde{j} \in \mathcal{P}_m, \quad \tilde{j} \equiv (j_1, \ldots, j_{i-1}, j_i, \ldots, j_{m-1}, m) \Rightarrow \text{sgn}(\tilde{j}) = \text{sgn}(j).
\]

\[
j' \in \mathcal{P}_m, \quad j' \equiv (j_1, \ldots, j_{i-1}, m, j_i, \ldots, j_{m-1}) \Rightarrow (-1)^{m-i} \text{sgn}(j') = \text{sgn}(\tilde{j}).
\]

For each \( i \), let \( j' = (j'_1, \ldots, j'_{m}) \equiv (j_1, \ldots, j_{i-1}, m, j_i, \ldots, j_{m-1}) \) and observe that the double sum \( \sum_{i=1}^{m} \sum_{j \in \mathcal{P}_{m-1}} \) is just a particular arrangement of \( \sum_{j' \in \mathcal{P}_m} \). Therefore

\[
\sum_{i=1}^{m} a_{i,m} (-1)^{i+m} \det(M_{i,m}) = \sum_{j' \in \mathcal{P}_m} (-1)^{2m} \text{sgn}(j') a_{1,j'_1} \cdots a_{m,j'_{m}},
\]

\[
= \sum_{j' \in \mathcal{P}_m} \text{sgn}(j') a_{1,j'_1} \cdots a_{m,j'_{m}} = \det(A),
\]

and the derivation is complete.

The Swiss mathematician Gabriel Cramer was the first to publish the relation between the solution of a square linear system and associated determinants. His observation has come to be known as Cramer’s rule.

First let me state Cramer’s rule for finding an inverse matrix. Let \( C \) denote the cofactor matrix for an \( m \times m \) matrix \( A \). Each element of \( C \) is given by

\[
c_{i,j} \equiv (-1)^{i+j} \det(M_{i,j}) \quad \text{for each } 1 \leq i \leq m, \ 1 \leq j \leq m.
\]

Then, if \( \det(A) \neq 0 \), Cramer’s rule gives

\[
A^{-1} = \frac{1}{\det(A)} C^T.
\]

This formula should not in general be regarded as a practical tool for computing inverses since working out cofactors can be very time consuming. It is handy when \( A \) is small however. For example, consider the \( 2 \times 2 \) matrix \( A \) (assume \( \det(A) \neq 0 \))

\[
A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Rightarrow C = \begin{pmatrix} a_{2,2} & -a_{2,1} \\ -a_{1,2} & a_{1,1} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{2,2} & -a_{2,1} \\ -a_{1,2} & a_{1,1} \end{pmatrix}.
\]

Here, \( C^T \) is found by flipping the diagonal entries of \( A \) and then changing the signs of the off–diagonal terms. Many students will memorize this \( 2 \times 2 \) formula.
The derivation of Cramer’s inverse formula is really pretty easy. Recall how matrix multiplication and cofactors are defined

\[(AC^T)_{i,j} = \sum_{k=1}^{m} a_{i,k} c_{k,j}^T = \sum_{k=1}^{m} a_{i,k} c_{j,k} = \sum_{k=1}^{m} a_{i,k} (-1)^{i+k} \det(M_{j,k}).\]

Now return to the 'row' cofactor expansion formula for the determinant and notice

\[a_{i,k} (-1)^{i+k} \det(M_{j,k}) = \det(A'),\]

where the matrix \(A'\) is obtained from \(A\) by replacing its \(j\)th row by its \(i\)th row. Therefore, since a matrix that has two identical rows has determinant zero, see that when \(i \neq j\) we have \(\det(A') = 0\). When \(i = j\) clearly \(\det(A') = \det(A)\). In other words

\[(AC^T)_{i,j} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \Rightarrow AC^T = \det(A) I,\]

and from this we easily see that Cramer’s inverse formula is valid.

An alternate formulation of Cramer’s rule gives the solution of \(Ax = b\) in terms of determinants.

\[Ax = b \Rightarrow x = A^{-1} b = \frac{1}{\det(A)} C^T b \Rightarrow x_i = \frac{1}{\det(A)} \sum_{k=1}^{m} c_{k,i} b_k.\]

Notice above that \(\sum_{k=1}^{m} c_{k,i} b_k = \sum_{k=1}^{m} b_k (-1)^{k+i} \det(M_{k,i})\) is the cofactor expansion for the determinant of the matrix formed by replacing \(A\)’s \(i\)th column with the column vector \(b\). That is

\[Ax = b \Rightarrow x_i = \frac{\det(B_i)}{\det(A)},\]

where the matrix \(B_i\) is the same as \(A\) except its \(i\)th column is replaced by vector \(b\). For example

\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\
\Rightarrow x_1 = \frac{1}{4-6} \det \begin{pmatrix} 5 & 2 \\ 6 & 4 \end{pmatrix} = \frac{8}{-2} = -4, \ x_2 = \frac{1}{4-6} \det \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix} = \frac{-9}{-2} = 9/2.
\]

You might also want to consider memorizing this form of Cramer’s rule long term.

1. Use a cofactor expansion to evaluate the determinant of the following.

(a) \[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\]  
(b) \[\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\]
2. Use a cofactor expansion to evaluate the determinant of the following.

(a) \[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 3 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
1 & 1 & 2 & 2 \\
1 & 0 & 1 & 1 \\
2 & 1 & 3 & 1 \\
2 & 0 & 1 & 3
\end{pmatrix}
\]

3. Use Cramer’s rule to determine the inverse matrix for the following.

(a) \[
\begin{pmatrix}
2 & 2 & 0 \\
2 & 1 & 0 \\
1 & 1 & 2
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
2 & 0 & 1 \\
2 & 1 & 0 \\
1 & 1 & 2
\end{pmatrix}
\]

4. Use Cramer’s rule to solve the following for the unknown \( x \).

(a) \[
\begin{pmatrix}
1 & 2 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
3 \\
4
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
3 & 1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

5. Use Cramer’s rule to solve the following for the unknown \( x \).

(a) \[
\begin{pmatrix}
2 & 2 & 0 \\
2 & 1 & 0 \\
1 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
2 & 1 & 0 \\
2 & 1 & 2 \\
1 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

An eigenvalue \( \lambda \) for an \( m \times m \) matrix \( A \) is a specific scalar value (i.e. a number) such that

\[
\det(A - \lambda I) = 0.
\]

For example

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1.
\]

Even for real matrices, its eigenvalues may be complex numbers. For example

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i.
\]

When \( A \) is \( m \times m \), refer back to the Leibniz formula for the determinant to see that

\[
\det(A - \lambda I) = p_m(\lambda) \quad \text{where} \quad p_m(\lambda) \quad \text{is a degree} \; m \quad \text{polynomial in the variable} \; \lambda.
\]

Let me show you why. Define \( A_{\lambda} \equiv A - \lambda I \) and use Leibniz to write

\[
\det(A_{\lambda}) = \sum_{j \in \mathcal{P}_m} \text{sgn}(j) (A_{\lambda})_{1,j_1} \cdots (A_{\lambda})_{m,j_m}
\]

\[
= \text{sgn}(1,2,\ldots,m)(A_{\lambda})_{1,1} \cdots (A_{\lambda})_{m,m} + \sum_{j \in \mathcal{P}_m \setminus \{1,2,\ldots,m\}} \text{sgn}(j) (A_{\lambda})_{1,j_1} \cdots (A_{\lambda})_{m,j_m}
\]

\[
= (a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) + \sum_{j \in \mathcal{P}_m \setminus \{1,2,\ldots,m\}} \text{sgn}(j) (A_{\lambda})_{1,j_1} \cdots (A_{\lambda})_{m,j_m}.
\]
For permutations \( j \neq (1, 2, \ldots, m) \) we can have \( j_i = i \) at most \( m - 2 \) times. This says the product in the right hand sum can contain no more than \( m - 2 \) diagonal terms of \( A_\lambda \). Therefore

\[
\det(A_\lambda) = (a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) + q_{m-2}(\lambda),
\]

where \( q_{m-2}(\lambda) \) is a polynomial which has degree no larger than \( m - 2 \). Furthermore, it’s easy to see

\[
(a_{1,1} - \lambda) \cdots (a_{m,m} - \lambda) = (-1)^m \left( \lambda^m - (a_{1,1} + \cdots + a_{m,m}) \lambda^{m-1} + \cdots \right)
\]

and so insert this into above to conclude

\[
\det(A - \lambda I) \equiv \det(A_\lambda) = \pm \left( \lambda^m - (a_{1,1} + \cdots + a_{m,m}) \lambda^{m-1} \right) + \tilde{q}_{m-2}(\lambda),
\]

where \( \tilde{q}_{m-2}(\lambda) \) denotes some other polynomial with degree no larger than \( m - 2 \).

\[\det(A - \lambda I) \equiv p_m(\lambda) \] is called \( A \)'s characteristic polynomial. To determine \( A \)'s eigenvalues, we must therefore find all roots to its degree \( m \) characteristic polynomial.

When \( m = 2 \), you’ll need to know the quadratic formula. For example

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{pmatrix} = \lambda^2 - 5\lambda - 2 = 0
\]

\[
\Rightarrow \lambda = \frac{5 \pm \sqrt{33}}{2}.
\]

When \( m \geq 3 \), you’ll need to rely on me to give you a problem where you either "luck-out" and factor the characteristic polynomial directly or in the "worst–case" guess some of the characteristic roots in order to reduce to a quadratic by long division.

Here’s a \( 3 \times 3 \) example. Suppose

\[
A = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix}.
\]

Cofactor \( A - \lambda I \) along, for example, the first row to get

\[
\det(A - \lambda I) = (3 - \lambda)(-1)^{1+1} \det \begin{pmatrix} 4 - \lambda & 2 \\ 1 & 5 - \lambda \end{pmatrix} - 1(-1)^{1+2} \det \begin{pmatrix} -2 & 2 \\ -1 & 5 - \lambda \end{pmatrix} + 1(-1)^{1+3} \det \begin{pmatrix} -2 & 4 - \lambda \\ -1 & 1 \end{pmatrix}
\]

\[
= (3 - \lambda) \left( (4 - \lambda)(5 - \lambda) - 2 \right) + \left( -2(5 - \lambda) + 2 \right) + \left( -2 + (4 - \lambda) \right)
\]

\[
= (3 - \lambda) \left( (4 - \lambda)(5 - \lambda) - 2 \right) + (2\lambda - 8) + (2 - \lambda)
\]

\[
= (3 - \lambda)(4 - \lambda)(5 - \lambda) + 3(\lambda - 4).
\]

Yay! \((\lambda - 4)\) factors out. So,

\[
\det(A - \lambda I) = -(\lambda - 4) \left( (\lambda - 3)(\lambda - 5) - 3 \right) = -(\lambda - 4) \left( \lambda^2 - 8\lambda + 12 \right).
\]
You can tell this is a homework problem because
\[
\det(A - \lambda I) = -(\lambda - 4)(\lambda - 2)(\lambda - 6) = 0 \implies \lambda = 2, 4, 6.
\]

6. Determine all eigenvalues for each of the following matrices.
(a) \[
\begin{pmatrix}
0 & -1 \\
2 & 3
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
0 & 2 \\
-1 & 3
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
1 & 2 \\
-1 & 4
\end{pmatrix}
\]
(d) \[
\begin{pmatrix}
-2 & -2 \\
6 & 5
\end{pmatrix}
\]
My answers: (a) \(\lambda = 1, 2\). (b) \(\lambda = 1, 2\). (c) \(\lambda = 2, 3\). (d) \(\lambda = 1, 2\).

7. Determine all eigenvalues for each of the following matrices.
(a) \[
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
0 & -1 & -1 \\
2 & 3 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]
(d) \[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -2 \\
2 & 2 & 4
\end{pmatrix}
\]
My answers: (a) \(\lambda = (3 \pm \sqrt{17})/2\). (b) \(\lambda = 1 \pm i\). (c) \(\lambda = 1, 2, 3\). (d) \(\lambda = 1, 2, 3\).