Eigenvectors and Constant Coefficient Systems of Differential Equations

Let $A$ denote an $m \times m$ matrix. An eigenvalue, $\lambda_*$, is a root to $A$’s characteristic polynomial, $p_m(\lambda)$. That is

$$\det(A - \lambda I) \equiv p_m(\lambda), \quad p_m(\lambda_*) = 0.$$  

In general, an eigenvalue may be a complex number. An eigenvector associated to an eigenvalue $\lambda_*$ is a nonzero vector, say $r_*$ which satisfies

$$Ar_* = \lambda_* r_* \iff (A - \lambda_* I)r_* = 0.$$  

Every eigenvalue has at least one associated eigenvector. If the algebraic multiplicity of the characteristic root $\lambda_*$ is greater than one however, it may or may not have more than one linearly independent eigenvectors. This subtlety is illustrated below.

Here are three examples.

For the first example, let

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

and compute

$$\det(A - \lambda I) = (3 - \lambda)^2 - 1 = (\lambda - 2)(\lambda - 4).$$

Therefore, this matrix has eigenvalues $\lambda = 2$ and $\lambda = 4$. To determine an eigenvector associated to eigenvalue $\lambda = 2$, find a nonzero solution to $(A - 2I)r = 0$ by eliminating the augmented matrix to echelon form

$$[A - 2I] = \begin{bmatrix} 3 - 2 & -1 \\ -1 & 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$  

From this back substitution gives

$$r_2 = \alpha \quad r_1 = r_2 = \alpha \quad \Rightarrow \quad r = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

We can take the free variable $\alpha$ to be any nonzero number. For ease, let’s take $\alpha = 1$. This gives us the first eigenvalue–eigenvector pair

$$\lambda = 2, \quad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

To determine an eigenvector associated to eigenvalue $\lambda = 4$, find a nonzero solution to $(A - 4I)r = 0$ by eliminating

$$[A - 4I] = \begin{bmatrix} 3 - 4 & -1 \\ -1 & 3 - 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$  

This time back substitution gives

$$r_2 = \alpha \quad r_1 = -r_2 = -\alpha \quad \Rightarrow \quad r = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$
Again take $\alpha = 1$ to find the second eigenvalue–eigenvector pair

$$\lambda = 4, \ r = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Here’s a second example. Let

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

and compute $\det(A - \lambda I) = (3 - \lambda)(1 - \lambda) + 1 = (\lambda - 2)^2.$

$\lambda = 2$ is the only eigenvalue here, and it has multiplicity two. Let’s see how many independent eigenvectors it has.

$$[A - 2I] = \begin{bmatrix} 3 - 2 & 1 \\ -1 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Back substitution gives

$$r_2 = \alpha, \quad r_1 = -r_2 = -\alpha \quad \Rightarrow \quad r = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

So for this example, the multiplicity two eigenvalue, $\lambda = 2$, has only one independent eigenvector.

For the third example, let

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

and compute $\det(A - \lambda I) = (4 - \lambda)^3 + 2 - 3(4 - \lambda)$

$$= -(\lambda - 3)^2(\lambda - 6).$$

Factorizing the characteristic polynomial as I did above is not obvious. Observe that $w \equiv (4 - \lambda) = 1$ is one of its roots and then reduce to a quadratic by long division

$$(4 - \lambda)^3 + 2 - 3(4 - \lambda) = w^3 - 3w + 2 = (w - 1)(w^2 + w - 2) = (w - 1)^2(w + 2).$$

Now, here we have only two distinct eigenvalues, $\lambda = 3$ which has multiplicity two, and $\lambda = 6$ which is simple. Let’s first find the eigenvector associated to the simple eigenvalue, $\lambda = 6$.

$$[A - 6I] = \begin{bmatrix} 4 - 6 & 1 & 1 \\ 1 & 4 - 6 & 1 \\ 1 & 1 & 4 - 6 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Back substitution gives

$$r_3 = \alpha, \quad r_2 = r_3 = \alpha \quad \Rightarrow \quad r = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and so we find

$$\lambda = 6, \ r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
The multiplicity two eigenvalue, $\lambda = 3$, may or may not have two independent eigenvectors. Let’s see.

$$[A - 3I] = \begin{bmatrix} 4 - 3 & 1 & 1 \\ 1 & 4 - 3 & 1 \\ 1 & 1 & 4 - 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Back substitution gives

$$r_3 = \alpha \quad r_2 = \beta \quad r_1 = -r_2 - r_3 = -\beta - \alpha \quad \Rightarrow \quad r = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$ 

From this we find two independent eigenvectors both associated to the same eigenvalue $\lambda = 3$,

$$r_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$ 

An $m \times m$ matrix $A$ is called diagonalizable if there is an invertible matrix $S$ such that

$$S^{-1}AS = D \quad \text{where} \quad D_{i,j} = 0 \quad \text{for every} \quad i, j \quad \text{with} \quad i \neq j.$$ 

$D$ above is called a diagonal matrix.

Here’s an important result you should all know. An $m \times m$ matrix is diagonalizable if and only if it has $m$ linearly independent eigenvectors. In particular, $S$ is the matrix $R$ whose columns are $A$’s eigenvectors $r_1, \ldots, r_m$ and $D$ is the diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_m$ on the diagonal listed in the same order counting multiplicity. That is

$$R^{-1}AR = \Lambda \quad \text{where} \quad \Lambda_{i,j} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad \Lambda_{i,i} = \lambda_i.$$ 

In the first example above we computed

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \Rightarrow \quad \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \quad R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and you should verify that

$$R^{-1}AR = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \Lambda.$$ 

In the third example we computed

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow \quad \Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Notice the multiplicity two eigenvalue, $\lambda = 3$, appears twice in $\Lambda$. See exercise 1 below.

The second example did not have a basis of eigenvectors, therefore

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

is not diagonalizable.
Let’s consider a system of differential equations posed as an initial value problem

\[ \frac{d}{dt} u = A u, \quad u(0) = u_0, \]

where the vector \( u(t) \in \mathbb{R}^m \) and the matrix \( A \in \mathbb{R}^{m \times m} \) is constant. For example, consider the coupled set of ODEs

\[ \frac{du}{dt} = 3u - v, \quad u(0) = 1, \]
\[ \frac{dv}{dt} = -u + 3v, \quad v(0) = 2. \]

Notice the equations for \( u \) and \( v \) do not stand alone, rather the two differential equations are coupled together. This set can be written in matrix form as

\[ \frac{d}{dt} u = A u, \quad u(0) = u_0, \]

where \( u(t) = (u, v) \), \( u_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), \( A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \).

Now, I’m going to show you how to solve such coupled systems. Suppose the constant matrix \( A \) above can be diagonalized, that is \( R^{-1} A R = \Lambda \). Make the change of dependent variables, \( u = R v \), to write

\[ \frac{d}{dt} u = A u \Rightarrow R \frac{d}{dt} v = A R v \Rightarrow \frac{d}{dt} v = R^{-1} A R v \Rightarrow \frac{d}{dt} v = \Lambda v, \quad v(0) = R^{-1} u_0. \]

Because the matrix \( \Lambda \) is diagonal, the system is decoupled in the variable \( v \). That is,

\[ \frac{dv_i}{dt} = \lambda_i v_i \quad \text{for each} \quad 1 \leq i \leq m, \]

and each of these are just scalar first order linear ODEs we solved way back at the beginning of the semester. Solving these

\[ \frac{dv_i}{dt} = \lambda_i v_i \Rightarrow v_i(t) = e^{\lambda_i t} v_i(0) \Rightarrow u(t) = R e^{\Lambda t} R^{-1} u_0, \]

where \( e^{\Lambda t} \) denotes the diagonal matrix with diagonal entries \( e^{\lambda_1 t}, \ldots, e^{\lambda_m t} \).

Let’s go back and solve the \( 2 \times 2 \) example above. Recall from earlier

\[ A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \]

and so we get

\[ \begin{pmatrix} u \\ v \end{pmatrix} = R e^{\Lambda t} R^{-1} u_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{2t} - e^{4t} \\ 3e^{2t} + e^{4t} \end{pmatrix}. \]

I suggest you take these, \( u(t) = \frac{1}{2} (3e^{2t} - e^{4t}) \), \( v(t) = \frac{1}{2} (3e^{2t} + e^{4t}) \), and plug them back into the original coupled set of ODEs to confirm they really are its solution.
1. Recall the third eigenvector example above.

\[
A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Compute \(R^{-1}\) and then verify \(R^{-1}AR = \Lambda\).

2. You computed the eigenvalues for the following matrices on your previous homework. Determine their eigenvectors and diagonalize if possible.

(a) \(\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\)  
(b) \(\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\)  
(c) \(\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\)  
(d) \(\begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}\)

Recall: (a) \(\lambda = 1, 2\). (b) \(\lambda = 1, 2\). (c) \(\lambda = 2, 3\). (d) \(\lambda = 1, 2\).

3. Do the same as in the previous exercise.

(a) \(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\)  
(b) \(\begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\)  
(c) \(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix}\)

Recall: (a) \(\lambda = 1 \pm i\). (b) \(\lambda = 1, 2, 3\). (c) \(\lambda = 1, 2, 3\).

In the next three exercises, use your work from 2 and 3 to solve the given coupled system.

4. \[
\frac{du}{dt} = u + 2v, \quad u(0) = 1, \quad \frac{dv}{dt} = -u + 4v, \quad v(0) = 2.
\]

5. \[
\frac{du}{dt} = u + v, \quad u(0) = 1, \quad \frac{dv}{dt} = -u + v, \quad v(0) = 2.
\]

6. \[
\frac{du}{dt} = u + w, \quad u(0) = 1, \quad \frac{dv}{dt} = v - 2w, \quad v(0) = 2, \quad \frac{dw}{dt} = 2u + 2v + 4w, \quad w(0) = 3.
\]

7. Solve the given coupled IVP by determining the eigenvalues and eigenvectors of the appropriate matrix.

\[
\frac{du}{dt} = 5u + 4w, \quad u(0) = 1, \quad \frac{dv}{dt} = 4v, \quad v(0) = 2, \quad \frac{dw}{dt} = 4u + 5w, \quad w(0) = 3.
\]