Laplace Transform Basics for ODEs

The Laplace transform provides an alternate method that, according to some, simplifies the process of analyzing and/or solving certain ODEs.

Let $u(t)$ denote a real valued function of a nonnegative real variable t. Its *Laplace transform*, denoted here by $L[u]$, is given by

$$(L) \qquad L[u](s) \equiv \int_0^\infty u(t)\, e^{-st}\, dt.$$

$L[u]$ signifies the function name of the Laplace transform of u, and (s) tells us $L[u]$ is a function of the so-called Laplace *dual variable* s. The variable s is generally considered to be a complex number. However, please note the following. $L[u]$ may or may not be well defined for a given real or complex s.

Let me show you by example what I mean. The calculus student is taught to treat the infinite integral as an improper integral

$$L[u](s) = \lim_{R\to\infty} \int_0^R u(t)\, e^{-st} dt.$$

Suppose $u(t) = e^t$, and compute

$$\int_0^R e^t\, e^{-st} dt = \int_0^R e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t}\Big|_{t=0}^{t=R} = \frac{1}{1-s}\left(e^{(1-s)R} - 1\right).$$

Clearly $\lim_{R\to\infty} e^{(1-s)R}$ exists only when the real part of the complex number s is greater than one, and so:

$$\text{For } \mathrm{Re}(s) > 1 \text{ we have } \; L[e^t](s) = \lim_{R\to\infty} \frac{1}{1-s}\left(e^{(1-s)R} - 1\right) = \frac{1}{s-1}.$$

Only functions which obey a certain exponential growth condition at infinity can have a Laplace transform. For example, $L[e^{t^2}](s)$ does not exist for any s.

It's very often the case that a given function is uniquely determined (at least almost everywhere) by its Laplace transform. In fact, there is an inverse transform formula, $L^{-1}[v]$, given by

$$(L^{-1}) \qquad L^{-1}[v](t) \equiv \frac{1}{2\pi i} \int_{\Gamma_{x_*}} v(s)\, e^{st}\, ds \quad \Rightarrow \quad u(t) = L^{-1}[L[u]](t),$$

where Γ_{x_*} denotes a path in the complex plane parameterized by $s(y) = x_* + iy$ with x_* some fixed (but large enough) value, $-\infty < y < \infty$ and $ds = i\, dy$. (Γ_{x_*} is a vertical line in the complex plane, $\mathrm{Re}(s) = x_*$.) Requisite techniques needed to evaluate the complex integral formula above for L^{-1} is well outside the scope of this class. If interested however, take a solid course in complex variables where topics such as *analytic continuation* and *the calculus of residues* are treated in some detail.

What I'm going to do here is give you a table of certain functions with their corresponding Laplace transforms. From these, together with some tricks, you'll be able to compute $L^{-1}[v(s)]$ for a fairly wide variety of $v(s) = L[u](s)$ without ever needing to evaluate the integral formula for L^{-1} directly.

	$u(t)$	$L[u](s)$		$u(t)$	$L[u](s)$
(1)	1	$\frac{1}{s}$	(2)	t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, \ldots$
(3)	e^{at}	$\frac{1}{s-a}$	(4)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
(5)	$\sin(bt)$	$\frac{b}{s^2+b^2}$	(6)	$\cos(bt)$	$\frac{s}{s^2+b^2}$
(7)	$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$	(8)	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
(9)	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	(10)	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
(11)	$\sinh(bt)$	$\frac{b}{s^2-b^2}$	(12)	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
(13)	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	(14)	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$

In the list above I made no attempt to spell out the set of complex s's where the Laplace transform integral is well defined. As it turns out, even when considering the path Γ_{x_*} needed to define the inverse formula L^{-1}, doing so is not really necessary. One only needs to regard $L[u](s)$ as defined on the whole complex plane, note its singularities (poles) as given, and then e.g. make sure the path Γ_{x_*} lies to their right.

Look back at the definition of $L[u](s)$ to see that it's a linear operator. That is, for functions $u_1(t)$ and $u_2(t)$ and constants c_1 and c_2 we have

$$L[c_1u_1 + c_2u_2](s) = c_1L[u_1](s) + c_2L[u_2](s).$$

This very important fact can be used, for example, to derive item (13) in the table above from item (3).

$$L[e^{at}\sinh(bt)] = L[e^{at}\tfrac{1}{2}(e^{bt} - e^{-bt})] = \frac{1}{2}\Big(L[e^{(a+b)t}] - L[e^{(a-b)t}]\Big)$$

$$\Rightarrow \quad L[e^{at}\sinh(bt)](s) = \frac{1}{2}\left(\frac{1}{s-(a+b)} - \frac{1}{s-(a-b)}\right) = \frac{b}{(s-a)^2-b^2}.$$

The table can also be used to determine $L^{-1}[v](t)$ for certain $v(s)$.

For example, suppose

$$v(s) = \frac{5s+6}{s^2+2s+3}.$$

Complete the square in the denominator to write

$$v(s) = \frac{5s+6}{s^2+2s+1-1+3} = \frac{5s+6}{(s+1)^2+2}$$
$$= \frac{5(s+1)+6-5}{(s+1)^2+2} = 5\left(\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right) + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right).$$

From the table, items (9) and (10), find that

$$L[e^{-t}\sin(\sqrt{2}t)](s) = \frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}, \quad L[e^{-t}\cos(\sqrt{2}t)](s) = \frac{s+1}{(s+1)^2+(\sqrt{2})^2}.$$

Therefore,

$$u(t) = 5\,e^{-t}\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\,e^{-t}\sin(\sqrt{2}t) \quad \Longleftrightarrow \quad L[u](s) = v(s) = \frac{5s+6}{s^2+2s+3}.$$

Or, in other words, $L^{-1}[v](t) = e^{-t}\left(5\cos(\sqrt{2}t) + \sin(\sqrt{2}t)/\sqrt{2}\right)$.

Here's another example. Suppose

$$v(s) = \frac{1}{(s^2+1)^2}.$$

Some clever adding and subtracting gives

$$v(s) = \frac{\frac{1}{2}(1+s^2)+\frac{1}{2}(1-s^2)}{(s^2+1)^2} = \frac{1}{2}\left(\frac{1}{s^2+1}\right) - \frac{1}{2}\left(\frac{s^2-1}{(s^2+1)^2}\right).$$

From items (5) and (8) in the table find that

$$L[\sin(t)](s) = \frac{1}{s^2+1}, \quad L[t\cos(t)](s) = \frac{s^2-1}{(s^2+1)^2},$$

and so

$$u(t) = \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t) \quad \Longleftrightarrow \quad L[u](s) = v(s) = \frac{1}{(s^2+1)^2}.$$

Two more very useful formulae are found by integration by parts. The first is

$$L[du/dt](s) = \int_0^\infty \frac{du}{dt}\,e^{-st}\,dt$$
$$= u(t)\,e^{-st}\Big|_{t=0}^{t=\infty} - \int_0^\infty u(t)(-se^{-st})\,dt = -u(0) + sL[u](s),$$

where $\lim_{t\to\infty} u(t)\,e^{-st} = 0$ is assumed, and then this can be used to derive

$$L[d^2u/dt^2](s) = \int_0^\infty \frac{d^2u}{dt^2}\,e^{-st}\,dt$$
$$= -u'(0) + sL[du/dt](s) = -u'(0) - s\,u(0) + s^2L[u](s),$$

where $\lim_{t\to\infty} u'(t)\,e^{-st} = 0$ is also assumed. Add these to the table

(15) $L\left[\dfrac{du}{dt}\right](s) = sL[u](s) - u(0)$ (16) $L\left[\dfrac{d^2u}{dt^2}\right](s) = s^2L[u](s) - su(0) - u'(0)$

I'll show you now how to use (15) and (16) to solve second order constant coefficient IVPs. Here's the first example.

$$\frac{d^2u}{dt^2} - u = e^t, \quad u(0) = 2, \; u'(0) = 3.$$

Take the Laplace transform of the differential equation and use (16) and (3) from the table

$$L[d^2u/dt^2 - u] = L[e^t] \quad \Rightarrow \quad s^2L[u] - s\,u(0) - u'(0) - L[u] = \frac{1}{s-1}$$

$$\Rightarrow \quad L[u](s) = \frac{2s+3}{(s^2-1)} + \frac{1}{(s-1)(s^2-1)} \equiv v_h(s) + v_p(s).$$

So, in order to determine $u(t)$, we must compute $L^{-1}[v_h]$ and $L^{-1}[v_p]$. Let's write v_h as follows and then use (12) and (11) from the table.

$$v_h(s) = 2\left(\frac{s}{s^2-1}\right) + 3\left(\frac{1}{s^2-1}\right) \quad \Rightarrow \quad L^{-1}[v_h](t) = 2\cosh(t) + 3\sinh(t).$$

Inverting v_p is a little bit more involved. We'll use partial fractions

$$v_p(s) = \frac{1}{(s-1)(s^2-1)} = \frac{1}{(s-1)^2(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+1)}.$$

Make <u>sure</u> you understand why I needed to use the form of the partial fractions expansion as indicated above. Otherwise, partial fractions won't work! Solve for A, B and C as usual to get

$$A = -\frac{1}{4}, \quad B = \frac{1}{2}, \quad C = \frac{1}{4}.$$

So, use the partial fraction expansion together with (3) and (4) from the table

$$v_p(s) = \frac{1}{4}\left(\frac{1}{s+1} - \frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{(s-1)^2}\right)$$

$$\Rightarrow \quad L^{-1}[v_p](t) = -\tfrac{1}{2}\,\sinh(t) + \tfrac{1}{2}\,te^t.$$

Therefore, the solution to the IVP is

$$u(t) = L^{-1}[v_h](t) + L^{-1}[v_p](t) = 2\cosh(t) + 3\sinh(t) - \tfrac{1}{2}\,\sinh(t) + \tfrac{1}{2}\,te^t.$$

BTW. Could have solved this IVP in half the time using the method of guessing and/or Duhamel.

Now I'm going to derive what is known as the *convolution theorem* for the Laplace transform. Define

$$h(t) \equiv \int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

h is called the convolution of f and g. Compute h's Laplace transform

$$L[h](s) = \int_0^\infty h(t)\,e^{-st}\,dt = \int_0^\infty \left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right) e^{-st}\,dt.$$

Reverse the order of double integration, i.e. $\int_0^\infty \int_0^t (\cdots)\, d\tau\, dt = \int_0^\infty \int_\tau^\infty (\cdots)\, dt\, d\tau$

$$L[h](s) = \int_0^\infty \int_\tau^\infty f(t-\tau)\, g(\tau)\, e^{-st}\, dt\, d\tau = \int_0^\infty \left(\int_\tau^\infty f(t-\tau) e^{-st}\, dt \right) g(\tau)\, d\tau.$$

(Again, don't worry about this step if you haven't taken Calculus 3 yet.) Change variable, $t - \tau \to t'$, in the inner integral

$$L[h](s) = \int_0^\infty \left(\int_0^\infty f(t') e^{-s(t'+\tau)}\, dt' \right) g(\tau)\, d\tau = \int_0^\infty \left(\int_0^\infty f(t')\, e^{-st'}\, dt' \right) g(\tau)\, e^{-s\tau}\, d\tau.$$

So finally, we have

$$L[h](s) = \left(\int_0^\infty f(t')\, e^{-st'}\, dt' \right) \left(\int_0^\infty g(\tau)\, e^{-s\tau}\, d\tau \right) = L[f](s)\, L[g](s).$$

This is the convolution theorem which states: The Laplace transform of the convolution is the product of the Laplace transforms.

Here's an example of the convolution theorem in action. Consider the following IVP with a given but arbitrary right hand side $f(t)$.

$$\frac{d^2u}{dt^2} + u = f(t), \quad u(0) = u'(0) = 0.$$

Take the Laplace transform of the differential equation and use (16) to find

$$s^2 L[u] + L[u] = L[f] \quad \Rightarrow \quad L[u](s) = \frac{1}{s^2+1}\, L[f](s).$$

Item (5) in the table says $L[\sin(t)](s) = 1/(s^2+1)$. So, from the convolution theorem

$$L[u](s) = L[\sin(t)](s)\, L[f](s) \quad \Rightarrow \quad u(t) = \int_0^t \sin(t-\tau)\, f(\tau)\, d\tau.$$

This is exactly the same formula obtained on your previous homework using Duhamel.

The last example can be generalized. Consider

$$\frac{d^2u}{dt^2} + a\,\frac{du}{dt} + bu = f(t), \quad u(0) = u'(0) = 0,$$

where a and b are constant. This implies

$$L[u](s) = \frac{1}{s^2 + as + b}\, L[f](s).$$

Complete the square to write

$$v_*(s) \equiv \frac{1}{s^2 + as + b} = \frac{1}{(s+\alpha)^2 - d},$$

where $\alpha = a/2$ and $d = \alpha^2 - b$. Let $\beta^2 = |d|$ and refer back to the table, items (4), (9) and (13), to see that

$$L[u_*](s) = v_*(s) \quad \Longleftrightarrow \quad u_*(t) = \begin{cases} e^{-\alpha t}\, t & \text{if } d = 0 \\ e^{-\alpha t} \sinh(\beta t)/\beta & \text{if } d > 0 \\ e^{-\alpha t} \sin(\beta t)/\beta & \text{if } d < 0. \end{cases}$$

As expected, $u_*(t)$ is the solution to the homogeneous IVP $u_*'' + au_*' + bu_* = 0$ with initial conditions $u_*(0) = 0$ and $u_*'(0) = 1$. So, exactly as seen on your previous homework, the solution to the original inhomogeneous problem, found here by the Laplace transform method, is

$$u(t) = \int_0^t u_*(t-\tau)\, f(\tau)\, d\tau.$$

Here's the last example. A model for a certain LC electrical circuit is given by

$$\frac{d^2q}{dt^2} + q = V(t), \quad q(0) = q'(0) = 0,$$

where $V(t)$ is a variable (nondimensional) supplied voltage. Suppose the voltage is a positive constant V_* up to $t = t_*$ and then suddenly removed. That is

$$V(t) = \begin{cases} V_* & \text{when } t < t_* \\ 0 & \text{when } t \geq t_*. \end{cases}$$

The formula derived in the last example says

$$q(t) = \int_0^t \sin(t-\tau)\, V(\tau)\, d\tau.$$

Since $V(\tau)$ is zero when $\tau \geq t_*$ and constant V_* when $\tau < t_*$ we find

$$q(t) = V_* \int_0^{\min(t,t_*)} \sin(t-\tau)\, d\tau = V_* \begin{cases} 1 - \cos(t) & \text{when } t < t_* \\ \cos(t-t_*) - \cos(t) & \text{when } t \geq t_*. \end{cases}$$

It's interesting to note the following about this solution. Both $q(t)$ and its derivative $q'(t)$ are defined and continuous, even at the time t_* when the voltage is discontinuously removed. However, q's second derivative can not be continuous at $t = t_*$ (look at the differential equation) and in fact fails to even exist at t_*.