Here we will set-up and solve the famous Bernoulli "what is the shape of a hanging chain" problem.

Let \( y(x) \geq 0 \) denote the height of the chain above the horizontal \( x \)-axis. For convenience, set coordinates so that \( y(0) = 0 \) and \( y'(0) = 0 \). The chain’s unit length tangent vector (pointing in the positive \( x \) direction) is given by

\[
t(x) = \frac{1}{\sqrt{1 + (y'(x))^2}} \begin{pmatrix} 1 \\ y'(x) \end{pmatrix}.
\]

There are two types of forces acting on chain elements: the internal tension force which acts tangentially to the chain, and the downward body force which is due to gravity. Consider a typical element from the point \((x_l, y(x_l))\) to the point \((x_r, y(x_r))\). The body force on this element is given by

\[
f_b = \rho \int_{x_l}^{x_r} \sqrt{1 + (y'(&0))^2} \, d\xi \left( 0 \right)
\]

where \( \rho \) denotes the mass per unit length of the chain, the integral is the length of the chain element, and \( g \) is the acceleration due to gravity. The internal tension force, which acts at the two ends of the element, is given by

\[
f_i = \kappa(x_r) \, t(x_r) - \kappa(x_l) \, t(x_l),
\]

where the scalar \( \kappa(x) \) denotes the magnitude of the tension force as a function of \( x \). We assume the chain is not in motion, therefore the sum of the forces on each element must balance out

\[
f_i + f_b = 0 \quad \Rightarrow \quad \kappa(x_r) - \kappa(x_l) = 0
\]

\[
\kappa(x_r) y'(x_r) - \kappa(x_l) y'(x_l) - \rho g \int_{x_l}^{x_r} \sqrt{1 + (y'(&0))^2} \, d\xi = 0,
\]

where we used the notation \( \kappa(x) \equiv \kappa(x)/\sqrt{1 + (y'(0))^2} \) to save space. Since \( \kappa(x_r) - \kappa(x_l) = 0 \) for arbitrary \( x_l \) and \( x_r \), we conclude \( \kappa(x) = \kappa(0) \) for every \( x \) along the chain. Also, since \( y'(0) = 0 \), we can write the constant \( \kappa(0) = \kappa(0) \equiv \kappa_0 \). In the second equation displayed on the right hand side above, set \( x_l = x \) and \( x_r = x + \Delta x \) to see

\[
\frac{\kappa_0}{\Delta x} \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left( \kappa(x + \Delta x) y'(x + \Delta x) - \kappa(x) y'(x) \right)
\]

\[
= \rho g \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x + \Delta x} \sqrt{1 + (y'(&0))^2} \, d\xi = \rho g \sqrt{1 + y'^2}.
\]

Finally, use this result, set \( z(x) = y'(x) \), and recall again that \( z(0) = y'(0) = 0 \), to get the separable IVP

\[
\frac{dz}{dx} = \gamma \sqrt{1 + z^2}, \quad z(0) = 0,
\]

where the constant \( \gamma = \rho g / \kappa_0 \).
Now let’s solve the IVP for $y(x)$.

$$\int \frac{dz}{\sqrt{1 + z^2}} = \int \gamma \, dx = \gamma (x + c).$$

But since $\cosh^2 \theta - \sinh^2 \theta = 1$, we use the substitution $z = \sinh \theta$, $dz = \cosh \theta \, d\theta$ to evaluate the left hand side integral

$$\int \frac{dz}{\sqrt{1 + z^2}} = \int \frac{\cosh \theta \, d\theta}{\sqrt{1 + \sinh^2 \theta}} = \int d\theta = \theta.$$

So

$$\theta = \gamma (x + c) \implies z = \sinh(\theta) = \sinh(\gamma (x + c)).$$

The initial condition $z(0) = 0$ shows $c = 0$. Therefore

$$z(x) = \sinh(\gamma x).$$

Finally, since $z = dy/dx$, integrate again

$$y(x) = \int \sinh(\gamma x) \, dx = \frac{1}{\gamma} (\cosh(\gamma x) + c),$$

and use $y(0) = 0$ to conclude that $c = -1$. Therefore

$$y(x) = \frac{1}{\gamma} (\cosh(\gamma x) - 1).$$

Can you figure out a way to determine the constant $\gamma$ knowing the the length of the chain and the distance between its attachment points? We’ll do this next.

Let $2L$ denote the length of the chain and $2R$ the distance between attachment points. Assume the chain is hanging symmetrically about the origin where it has length $L$ from the point $(0, y(0)) = (0, 0)$ to the point $(R, y(R))$. Using the arclength formula we find

$$L = \int_0^R \sqrt{1 + (y'(\xi))^2} \, d\xi = \int_0^R \frac{\cosh(\gamma \xi) \, d\xi}{\sqrt{1 + \sinh^2 \gamma \xi}} = \int_0^R \frac{\cosh(\gamma \xi) \, d\xi}{\gamma} = \frac{\sinh(\gamma R)}{\gamma}.$$

The function $\sinh(\theta)/\theta$ is a continuous one to one mapping from the interval $(0, \infty)$ onto $(1, \infty)$. Therefore, for a given $L/R > 1$, there is a unique positive number $\theta_*$ so that

$$\frac{L}{R} = \frac{\sinh(\theta_*)}{\theta_*} \implies \gamma R = \theta_*.$$

This relation gives the sought for constant $\gamma$ which yields

$$y(x) = \frac{R}{\theta_*} (\cosh(\theta_* x/R) - 1).$$

It is interesting to note that since $\theta_*$ depends only on the ratio $L/R$, the hanging chain’s shape function, $y(x)$, depends only on the parameters $R$ and $L/R$. That is, the shape of a hanging chain does not depend on its density or the acceleration of gravity.