Math 3331. The Hanging Chain Problem

Here we will set-up and solve the famous Bernoulli "what is the shape of a hanging chain" problem.

Let $y(x) \geq 0$ denote the height of the chain above the horizontal x-axis. For convenience, set coordinates so that $y(0) = 0$ and $y'(0) = 0$. The chain's unit length tangent vector (pointing in the positive x direction) is given by

$$\mathbf{t}(x) = \frac{1}{\sqrt{1 + (y'(x))^2}} \begin{pmatrix} 1 \\ y'(x) \end{pmatrix}.$$

There are two types of forces acting on chain elements: the internal tension force which acts tangentially to the chain, and the downward body force which is due to gravity. Consider a typical element from the point $(x_l, y(x_l))$ to the point $(x_r, y(x_r))$. The body force on this element is given by

$$\mathbf{f}_b = \rho \int_{x_l}^{x_r} \sqrt{1 + (y'(\xi))^2}\, d\xi \begin{pmatrix} 0 \\ -g \end{pmatrix},$$

where ρ denotes the mass per unit length of the chain, the integral is the length of the chain element, and g is the acceleration due to gravity. The internal tension force, which acts at the two ends of the element, is given by

$$\mathbf{f}_i = \kappa(x_r)\, \mathbf{t}(x_r) - \kappa(x_l)\, \mathbf{t}(x_l),$$

where the scalar $\kappa(x)$ denotes the magnitude of the tension force as a function of x. We assume the chain is not in motion, therefore the sum of the forces on each element must balance out

$$\mathbf{f}_i + \mathbf{f}_b = \mathbf{0} \quad \Rightarrow \quad \begin{array}{l} \tilde{\kappa}(x_r) - \tilde{\kappa}(x_l) = 0 \\ \tilde{\kappa}(x_r) y'(x_r) - \tilde{\kappa}(x_l) y'(x_l) - \rho g \displaystyle\int_{x_l}^{x_r} \sqrt{1 + (y'(\xi))^2}\, d\xi = 0, \end{array}$$

where we used the notation $\tilde{\kappa}(x) \equiv \kappa(x)/\sqrt{1 + (y'(x))^2}$ to save space. Since $\tilde{\kappa}(x_r) - \tilde{\kappa}(x_l) = 0$ for arbitrary x_l and x_r, we conclude $\tilde{\kappa}(x) = \tilde{\kappa}(0)$ for every x along the chain. Also, since $y'(0) = 0$, we can write the constant $\tilde{\kappa}(0) = \kappa(0) \equiv \kappa_0$. In the second equation displayed on the right hand side above, set $x_l = x$ and $x_r = x + \Delta x$ to see

$$\begin{aligned} \kappa_0 \frac{dy'}{dx} &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Big(\tilde{\kappa}(x + \Delta x)\, y'(x + \Delta x) - \tilde{\kappa}(x)\, y'(x) \Big) \\ &= \rho g \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} \sqrt{1 + y'(\xi)^2}\, d\xi = \rho g \sqrt{1 + y'^2}. \end{aligned}$$

Finally, use this result, set $z(x) = y'(x)$, and recall again that $z(0) = y'(0) = 0$, to get the separable IVP

$$\frac{dz}{dx} = \gamma \sqrt{1 + z^2}, \quad z(0) = 0, \quad \text{where the constant } \gamma = \rho g / \kappa_0.$$

Now let's solve the IVP for $y(x)$.

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \gamma\, dx = \gamma\,(x+c).$$

But since $\cosh^2\theta - \sinh^2\theta = 1$, we use the substitution $z = \sinh\theta$, $dz = \cosh\theta\, d\theta$ to evaluate the left hand side integral

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\cosh\theta\, d\theta}{\sqrt{1+\sinh^2\theta}} = \int d\theta = \theta.$$

So

$$\theta = \gamma\,(x+c) \quad \Rightarrow \quad z = \sinh(\theta) = \sinh(\gamma\,(x+c)).$$

The initial condition $z(0)=0$ shows $c=0$. Therefore

$$z(x) = \sinh(\gamma\, x).$$

Finally, since $z = dy/dx$, integrate again

$$y(x) = \int \sinh(\gamma x)\, dx = \frac{1}{\gamma}(\cosh(\gamma\, x) + c),$$

and use $y(0)=0$ to conclude that $c=-1$. Therefore

$$y(x) = \frac{1}{\gamma}(\cosh(\gamma\, x) - 1).$$

Can you figure out a way to determine the constant γ knowing the the length of the chain and the distance between its attachment points? We'll do this next.

Let $2L$ denote the length of the chain and $2R$ the distance between attachment points. Assume the chain is hanging symmetrically about the origin where it has length L from the point $(0, y(0)) = (0,0)$ to the point $(R, y(R))$. Using the arclength formula we find

$$L = \int_0^R \sqrt{1+(y'(\xi))^2}\, d\xi = \int_0^R \sqrt{1+\sinh^2(\gamma\,\xi)}\, d\xi = \int_0^R \cosh(\gamma\,\xi)\, d\xi = \frac{\sinh(\gamma\, R)}{\gamma}.$$

The function $\sinh(\theta)/\theta$ is a continuous one to one mapping from the interval $(0,\infty)$ onto $(1,\infty)$. Therefore, for a given $L/R > 1$, there is a unique positive number θ_* so that

$$\frac{L}{R} = \frac{\sinh(\theta_*)}{\theta_*} \quad \Rightarrow \quad \gamma R = \theta_*.$$

This relation gives the sought for constant γ which yields

$$y(x) = \frac{R}{\theta_*}\left(\cosh(\theta_* x/R) - 1\right).$$

It is interesting to note that since θ_* depends only on the ratio L/R, the hanging chain's shape function, $y(x)$, depends only on the parameters R and L/R. That is, the shape of a hanging chain does not depend on its density or the acceleration of gravity.