Introduction to First Order Systems

We’ll look at systems of differential equations of the form
\[
\frac{d}{dt} u = f(u, t),
\]
where \( t \) is the independent variable, \( u \) is the dependent unknown vector
\[
u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix},
\]
and \( f \) is a given vector function of \( u \) and \( t \)
\[
f(u, t) = \begin{pmatrix} f_1(u_1, \ldots, u_n, t) \\ \vdots \\ f_n(u_1, \ldots, u_n, t) \end{pmatrix}.
\]

A classic example of a first order system is the *Lotka-Volterra* predator-prey model
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ uv - v \end{pmatrix},
\]
where \( u(t) \) represents the time dependent number density of a prey species, and \( v(t) \) represents the time dependent number density of a predator species.

In most applications, an ordinary differential equation is a system. Consider two particles, the first located at position \( x_1 \) having mass \( m_1 \) and the second at \( x_2 \) with mass \( m_2 \). According to Newton’s law of gravity, the force exerted on the first particle by the second particle is
\[
m_1 m_2 G \frac{x_2 - x_1}{||x_2 - x_1||^3},
\]
\( (G \) is the universal gravitational constant) and the force exerted on the second particle by the first particle is
\[
m_2 m_1 G \frac{x_1 - x_2}{||x_1 - x_2||^3}.
\]

Therefore, since force equals mass times acceleration
\[
\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G \frac{m_2 (x_2 - x_1)}{||x_2 - x_1||^3} \frac{m_1 (x_1 - x_2)}{||x_1 - x_2||^3} \equiv \begin{pmatrix} a_{1,2} \\ a_{2,1} \end{pmatrix}.
\]

In three space dimensions, this is a six component second order system. However, it can be transformed into a twelve component first order system by noting that the velocity \( v \) of each particle is given by \( dx/dt \), thus yielding
\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ a_{1,2} \\ v_2 \\ a_{2,1} \end{pmatrix} \equiv f(x_1, v_1, x_2, v_2).
\]
The two-particle example above illustrates a general principle. An \( n^{th} \) order differential equation can be rewritten as a first order system. For example, the third order scalar equation
\[
\frac{d^3 u}{dt^3} + \left( \frac{du}{dt} \right)^2 + \sin(u) = 0,
\]
can be rewritten as the following first order system
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \begin{pmatrix} v \\ w \\ -v^2 - \sin(u) \end{pmatrix}
\end{align*}
\]
by substituting \( du/dt \equiv v \) and \( d^2u/dt^2 = dv/dt \equiv w \).

Here are some exercises.

1. Write the scalar differential equations as first order systems.
   \( \text{(a) } \frac{d^2 u}{dt^2} + u = 0 \)
   \( \text{(b) } \frac{d^2 u}{dt^2} - 2 \frac{du}{dt} + u = 0 \)

2. Write as first order systems.
   \( \text{(a) } \frac{d^2 u}{dt^2} + \frac{2}{t} \frac{du}{dt} + \frac{1}{t^2} u = e^t \)
   \( \text{(b) } \frac{d^2 u}{dt^2} - t^2 \frac{du}{dt} + t^4 u + \cos(t) = 0 \)

3. Write as first order systems.
   \( \text{(a) } \frac{d^3 u}{dt^3} - 2 \frac{d^2 u}{dt^2} - 3 \frac{du}{dt} - 4 u = 0 \)
   \( \text{(b) } \frac{d^4 u}{dt^4} - u = \sin(t) \)

4. Write as first order systems.
   \( \text{(a) } \frac{d^3 u}{dt^3} = (u^2 + 1)e^t \cos \left( \frac{d^2 u}{dt^2} \right) \)
   \( \text{(b) } \frac{d^2 u}{dt^2} = \sqrt{t^2 + \left( \frac{du}{dt} \right)^2} \)

5. Write the second order system as a first order system.
   \[
   \frac{d^2}{dt^2} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} v \\ -u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
   \]
I will call a first order system linear if it can be written in the form
\[
\frac{d}{dt} u + A(t) u = f(t),
\]
where \( A(t) \) is a given matrix (the term \( A(t) u \) denotes the usual matrix–vector multiplication), and \( f(t) \) is a given vector function. I’ll say the linear system is constant coefficient when \( A \) is a constant matrix. I’ll say the linear system is homogeneous when \( f \) is the zero vector. The first order system you obtained in exercise 1(a) can be written as
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
This is linear, constant coefficient and homogeneous. The first order system you obtained in exercise 2(a) can be written as
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1/t^2 & 2/t \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.
\]
This is linear, variable coefficient and inhomogeneous.

Consider a general \( n \)-dimensional linear, possibly variable coefficient, and homogeneous system
\[
\frac{d}{dt} u + A(t) u = 0.
\]
Suppose this system has \( n \) solution vectors denoted by \( u_{*,1}, \ldots, u_{*,n} \). (These are column vectors.) Form what is called the Wronskian matrix, \( W \), by inserting these \( n \) solution vectors into the matrix \( W \) column-wise. The Wronskian determinant, \( w \), is given by
\[
w = \det(W).
\]
Later, I will show that the Wronskian determinant solves the scalar differential equation
\[(\text{Wronski-E}) \quad \frac{dw}{dt} + \text{Tr}(A(t)) w = 0,\]
where \( \text{Tr}(A) \equiv a_{1,1} + \cdots + a_{n,n} \) denotes the trace of the \( n \times n \) matrix \( A \). We can explicitly solve this scalar first order linear equation, just like we did in the first couple of weeks of class, to find that
\[(\text{Wronski-S}) \quad w(t) = e^{-\int_{t_0}^{t} a(s) \, ds} \, w(t_0) \quad \text{where} \quad a(s) = \text{Tr}(A(s)).\]
Here’s what you want to take away from this. Assume \( \int_{t_0}^{t} \text{Tr}(A(s)) \, ds \) exists and is bounded for all \( t \) in some interval containing \( t_0 \). Then, for any \( t \) in that interval, the solution set of vectors \( \{u_{*,1}(t), \ldots, u_{*,n}(t)\} \) is a basis for \( \mathbb{R}^n \) if and only if it is a basis at time \( t = t_0 \).

We’ll verify the work in the previous paragraph by considering the specific system
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 2/t & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Here \( \text{Tr}(A(t)) = 0 + 2/t \), and so from (Wronski-E) and (Wronski-S)
\[
\frac{dw}{dt} + \frac{2}{t} w = 0 \quad \Rightarrow \quad w(t) = e^{-\int_{t_0}^{t} \frac{2}{s} \text{ds}} w(t_0) = (t_0^2/t^2) w(t_0).
\]

To verify this is correct, check that away from \( t = 0 \) the system has two explicit independent solutions
\[
\mathbf{u}_{*,1}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_{*,2}(t) = \begin{pmatrix} 1/t \\ -1/t^2 \end{pmatrix} \quad \Rightarrow \quad W(t) = \begin{pmatrix} 1 & 1/t \\ 0 & -1/t^2 \end{pmatrix},
\]
and calculate
\[
w(t) = \det(W(t)) = -1/t^2 \quad \Rightarrow \quad \frac{w(t)}{w(t_0)} = \frac{-1/t^2}{-1/t_0^2} = (t_0^2/t^2).
\]
This is in agreement with the (Wronski-S) calculation done above.

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Do the following for the next 3 exercises.

(a) Verify the given solutions are indeed solutions.

(b) Solve (Wronski-E) for \( w(t) \) in terms of \( w(0) \).

(c) Verify part (b) by computing the Wronskian determinant directly.

6. The linear system
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
has two solutions
\[
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}.
\]

7. The linear system
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
has two solutions
\[
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = e^{-t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.
\]

8. The linear system
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
has two solutions
\[
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = e^{-4t} \begin{pmatrix} 2t + 1 \\ t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = e^{-4t} \begin{pmatrix} 4t \\ 2t - 1 \end{pmatrix}.
\]
Lastly, I will derive the scalar equation (Wronski-E) for the linear and homogeneous $n \times n$ system
\[ \frac{d}{dt} \mathbf{u} + A(t) \mathbf{u} = 0. \]

Suppose we have $n$ solutions $\mathbf{u}_s, 1, \ldots, \mathbf{u}_s, n$ (recall these are column $n$–vectors) from which we define the $n \times n$ Wronskian matrix and its determinant
\[ W = ( \mathbf{u}_s, 1 \cdots \mathbf{u}_s, n ) \quad \text{and} \quad w = \det(W). \]

Use the row-wise product rule for determinants to obtain
\[ \frac{dw}{dt} = \sum_{k=1}^{n} \det(D_k) \quad \text{where} \quad D_k \equiv \begin{pmatrix} \mathbf{u}_{1,s} \\ \vdots \\ \mathbf{d}\mathbf{u}_{k,s}/dt \\ \vdots \\ \mathbf{u}_{n,s} \end{pmatrix}. \]

Above, $\mathbf{u}_{i,s}$ are row vectors and $D_k$ is formed by differentiating the $k^{th}$ row of $W$. This is why it’s called the product rule. Using the system
\[ \frac{d}{dt} \mathbf{u}_{s,j} + A(t) \mathbf{u}_{s,j} = 0 \]
\[ \Rightarrow \frac{d}{dt} u_{k,j} + \sum_{l=1}^{n} a_{k,l}(t) u_{l,j} = 0 \]
\[ \Rightarrow \frac{d}{dt} \mathbf{u}_{k,s} = -\sum_{l=1}^{n} a_{k,l}(t) \mathbf{u}_{l,s}. \]

Using this and the fact that the determinant is multi-linear we see
\[ D_k = \begin{pmatrix} \mathbf{u}_{1,s} \\ \vdots \\ -\sum_{l=1}^{n} a_{k,l}(t) \mathbf{u}_{l,s} \\ \vdots \\ \mathbf{u}_{n,s} \end{pmatrix} \]
\[ \Rightarrow \det(D_k) = -\sum_{l=1}^{n} a_{k,l}(t) \det(M_{k,l}) \quad \text{where} \quad M_{k,l} = \begin{pmatrix} \mathbf{u}_{1,s} \\ \vdots \\ \mathbf{u}_{l,s} \\ \vdots \\ \mathbf{u}_{n,s} \end{pmatrix}. \]

Note that the row vector $\mathbf{u}_{l,s}$ appears in the $k^{th}$ row of $M_{k,l}$. Therefore,
\[ \forall l \neq k \quad \det(M_{k,l}) = 0. \]
since in this case $M_{k,l}$ has two identical rows, and when $l = k$

$$\det(M_{k,k}) = \det(W) = w.$$ 

This shows

$$\det(D_k) = -a_{k,k}(t) \det(W) = -a_{k,k}(t) w,$$

from which the scalar equation (Wronski-E) easily follows.