First Order ODE Applications

1. A tank is initially filled with 20 gallons of fresh water. A salt brine solution, at one pound of salt per one gallon of water, is piped into the tank at one gallon per minute. There is another pipe which will output one gallon per minute of the fully mixed water/brine solution. Let $s(t)$ denote the amount of salt in the tank at time t measured in minutes. From what is given, $s(0) = 0$, and also

$$\left(\frac{ds}{dt}\right)_{in} = (1 \ lb/gal)(1 \ gal/min) = 1 \ lb/min$$

$$\left(\frac{ds}{dt}\right)_{out} = -(s(t)/20 \ lb/gal)(1 \ gal/min) = -s(t)/20 \ lb/min$$

$$\Rightarrow \quad \frac{ds}{dt} = 1 - s/20.$$

Determine $s(t)$ for $t > 0$.

2. In the previous exercise, 1 gal/min flowed in and 1 gal/min flowed out. Suppose we open up the input valve a bit so that now 2 gallons per minute flows in. Therefore, at time t there are $20 + t$ gallons of solution in the tank.

(a) What is $(ds/dt)_{in}$?   (b) What is $(ds/dt)_{out}$?

(c) What is $s(t)$ for $t > 0$?

3. Consider a cylindrical water tank with cross sectional area A_T. At the bottom of the tank there is a small hole with area A_h. One can show that the speed of the water flowing out the hole at the bottom is given by $\sqrt{2gz}$ where g is the acceleration of gravity and z is the depth of the water in the tank. (This fact comes from what is called *Bernoulli's principle* in fluid mechanics. See http://en.wikipedia.org/wiki/Bernoulli's_principle.) Therefore, the volume of water lost through the small hole per unit time (the rate of loss) is the the area of the hole times the speed of the water; i.e.

$$\left(\frac{dV}{dt}\right)_{out} = A_h\sqrt{2gz}.$$

The volume of water in the tank is $V = A_T z$. So we have a differential equation

$$\frac{dV}{dt} = -\left(\frac{dV}{dt}\right)_{out} \quad \Rightarrow \quad A_T\frac{dz}{dt} = -A_h\sqrt{2gz} \quad \Rightarrow \quad \frac{dz}{dt} = -k\sqrt{z},$$

where $k = \sqrt{2g}\,A_h/A_T$.

(a) Taking $z(0) = H$, how long does it take to drain the tank? (Answer: Solve $z(t_*) = 0$ to get $t_* = 2\sqrt{H}/k$.)

(b) Suppose we also pump water into the tank at a constant flow rate of I. What values

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of I will assure the tank never fully drains of water? (Hint: Don't explicitly solve for $z(t)$. Separate variables and use the implicit relation $z(t)$ satisfies.)

Let T denote the core temperature of a body which is immersed in a fluid with ambient temperature T_A. Newton's law of cooling (or heating) states that
$$\frac{dT}{dt} = k_b(T_A - T),$$
where the rate constant k_b is characteristic of the body itself and therefore must be somehow determined. (Clearly k_b is large for a mouse and much much smaller for an elephant.) See e.g. `http://en.wikipedia.org/wiki/Convective_heat_transfer`.

4. A man has been killed in his home. The police forensic team arrives at the crime scene at 10 am. They immediately measure the scene's ambient temperature, $T_A = 70°F$, which they assume is constant, and the dead man's body core temperature, $T(0) = 80°F$. One hour later they again measure the man's core temperature and got $T(1) = 75°F$. Assume the man's core temperature was $98°F$ at the moment he was killed. When was the crime committed? (Hint: Solve the ode with data given at $T(0)$, use this and $T(1)$ to determine k_b, which shows the dead man's body temperature is given by $T(t) = 10(1/2)^t + 70$. From this I'd say he was killed around 8:30 am.)

5. A large vat of water is outside, and its heat rate constant is measured to be $k_b = \log(1.1)$. The ambient temperature outside is not constant. In fact, over the course of many days we observe
$$T_A(t) = 70 + 10\sin(t\pi/12) \ °F,$$
where t is measured in hours starting from 10 am clock time. That is, the outside temperature sinusoidally ranges from a low of $60°F$ at 4 am to a high of $80°F$ at 4 pm.
(a) Suppose the water temperature is $T(0) = T_0$ initially. Use Newton's law of cooling to determine the water temperature $T(t)$ in the tank at time t? Answer: I got
$$T(t) = e^{-k_b t} T_0 + 70(1 - e^{-k_b t}) + \frac{10 k_b}{k_b^2 + \omega^2} \left(k_b \sin(\omega t) - \omega\cos(\omega t) + \omega e^{-k_b t} \right),$$
where $k_b = \log(1.1)$ and $\omega = \pi/12$.
(b) In large time, the terms above involving $e^{-k_b t}$ will damp out, and so then the tank's water temperature will approximately satisfy
$$T(t) \approx 70 + \frac{10 k_b}{k_b^2 + \omega^2} \left(k_b \sin(\omega t) - \omega\cos(\omega t) \right).$$
Show that this can be rewritten as
$$T(t) \approx 70 + \frac{10 k_b}{\sqrt{k_b^2 + \omega^2}} \sin(\omega t - \phi)$$

for some phase shift angle ϕ. From this, compute the water's *large time* minimum and maximum temperatures and at what wall clock times these two extreme temperatures are attained. (I got t_{min} corresponds to around 8:40 am and t_{max} to around 8:40 pm.)