Constant Coefficient Systems

---

1. Determine all eigenvalues for each of the following matrices.

$$\text{(a)} \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}$$

My answers: (a) $\lambda = 1, 2$. (b) $\lambda = 1, 2$. (c) $\lambda = 2, 3$. (d) $\lambda = 1, 2$.

2. Determine all eigenvalues for each of the following matrices.

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix}$$

My answers: (a) $\lambda = (3 \pm \sqrt{17})/2$. (b) $\lambda = 1 \pm i$. (c) $\lambda = 1, 2, 3$. (d) $\lambda = 1, 2, 3$.

3. Compute the eigenvectors for the matrices in exercises 1(a), 1(c) and 1(d), Let A denote the given matrix for each part. (1) Form the matrix R whose columns are made of the eigenvectors you just computed, (2) determine the inverse R^{-1} and finally (3) verify that $R^{-1}AR = \Lambda$ where Λ is the diagonal matrix composed of A's eigenvalues.

4. Do the same as in the previous exercise, this time however for the matrices given in 2(b), 2(c) and 2(d).

You learned about what are called *real analytic functions* in calculus; see power series in general and in particular see Taylor's theorem. For example, recall

$$(1 - at)^{-1} = \sum_{n=0}^{\infty} (at)^n \quad \text{has radius of convergence given by } |t| < 1/|a|,$$

and

$$e^{at} = \sum_{n=0}^{\infty} \frac{1}{n!}(at)^n \quad \text{has radius of convergence given by } |t| < \infty.$$

Two other important Taylor expansions you should all recall are

$$\sin(at) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(at)^{2n+1} \quad \text{and} \quad \cos(at) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(at)^{2n},$$

where both of these have an infinite radius of convergence.

Analytic functions can be defined for square matrices. For example, given a square matrix A, we can define

$$(I - At)^{-1} = \sum_{n=0}^{\infty} (At)^n \quad \text{which converges for any } t \text{ satisfying } |||A|||\,|t| < 1,$$

and

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}\,(At)^n \quad \text{which converges for any } t.$$

Note that above and throughout we define $A^0 \equiv I$, the identity matrix.

See that the infinite series representation for $(I - At)^{-1}$ makes sense by calculating

$$\begin{aligned}
(I - At)(I - At)^{-1} &= (I - At)\sum_{n=0}^{\infty}(At)^n \\
&= I\sum_{n=0}^{\infty}(At)^n - At\sum_{n=0}^{\infty}(At)^n = \sum_{n=0}^{\infty}(At)^n - \sum_{n=0}^{\infty}(At)^{n+1} \\
&= \sum_{n=0}^{\infty}(At)^n - \sum_{n=1}^{\infty}(At)^n = (At)^0 \equiv I.
\end{aligned}$$

That is, for sufficiently small $|t|$, the infinite series $\sum_{n=0}^{\infty}(At)^n$ really is the inverse matrix of $(I - At)$.

You should also carefully verify the following.

$$\frac{d}{dt}e^{At} = A\,e^{At} = e^{At}A.$$

Let's check that this is true by first calculating

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}\,(At)^n = \sum_{n=1}^{\infty}\frac{1}{n!}\,A^n\frac{d}{dt}t^n = \sum_{n=1}^{\infty}\frac{1}{n!}\,A^n n t^{n-1}.$$

(The lower limit in the sum was changed to $n = 1$ since the $n = 0$ term is constant.) This is a valid calculation because a power series can always be differentiated term-by-term inside its radius of convergence. Clearly $n/n! = 1/(n-1)!$, so we have shown

$$\begin{aligned}
\frac{d}{dt}e^{At} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\,A^n t^{n-1} &= A\left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\,A^{n-1}t^{n-1}\right) \\
\text{or} \quad &= \left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\,A^{n-1}t^{n-1}\right)A.
\end{aligned}$$

Reindex in the bracketed terms above, i.e. let $n - 1 \to n$, to see they are equal to e^{At}.

What this shows is well worth remembering, (especially for the next midterm exam and final). The constant coefficient, linear and homogeneous system

$$\frac{d}{dt}\mathbf{u} = A\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{is solved by} \quad \mathbf{u}(t) = e^{At}\mathbf{u}_0.$$

You will use this to solve several IVPs in the exercises below.

In general $e^{A+B} \neq e^A e^B$. However, it is true that

$$\text{when } AB = BA \text{ we do have } e^{A+B} = e^A e^B.$$

You can prove this (I won't here) by employing the *binomial theorem* for matrices:

$$\text{If } AB = BA, \text{ then } \ (A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k.$$

It therefore follows that since A and $-A$ commute

$$I = e^{A-A} = e^{A+(-A)} = e^A e^{-A} \quad \Rightarrow \quad (e^A)^{-1} = e^{-A}.$$

When A can be diagonalized, and f is real analytic, there is a particularly simple way to evaluate the matrix $f(A)$ in closed form, that is without infinite series. Suppose we can write

$$R^{-1}AR = \Lambda \quad \Rightarrow \quad A = R\Lambda R^{-1},$$

where Λ is a diagonal matrix composed of A's eigenvalues. Then $f(A)$ is given by

$$f(A) = Rf(\Lambda)R^{-1} \ \text{ where } \ f(\Lambda) = \text{diag}(f(\lambda_1), \ldots, f(\lambda_n)).$$

I'll derive the formula after giving a couple of examples. Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and compute} \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

From these, you can easily determine $\cos(A)$ in closed form.

$$\begin{aligned} \cos(A) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos(1) & 0 \\ 0 & \cos(3) \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \cos(1)+\cos(3) & \cos(3)-\cos(1) \\ \cos(3)-\cos(1) & \cos(1)+\cos(3) \end{pmatrix}. \end{aligned}$$

Similarly,

$$e^{At} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \cdots$$

Let me now derive the closed form expression for $f(A)$ when A is diagonalizable. Suppose $f(x)$ is given by the power series

$$f(x) = \sum_{k=0}^{\infty} \gamma_n x^k \quad \Rightarrow \quad f(A) \equiv \sum_{k=0}^{\infty} \gamma_k A^k$$

where $\gamma_k \in \mathbb{R}$ is f's kth Taylor coefficient. Since $A = R\Lambda R^{-1}$, see that

$$A^k = \left(R\Lambda R^{-1}\right)^k = \left(R\Lambda R^{-1}\right) \cdots \left(R\Lambda R^{-1}\right) = R\Lambda^k R^{-1}.$$

Therefore

$$f(A) = R\left(\sum_{k=0}^{\infty} \gamma_k \Lambda^k\right) R^{-1}.$$

Finally, since Λ is diagonal, $\Lambda^k = \text{diag}(\lambda_1^k, \ldots, \lambda_n^k)$, which shows

$$f(A) = R\left(\text{diag}(\sum_{k=0}^{\infty} \gamma_k \lambda_1^k, \ldots, \sum_{k=0}^{\infty} \gamma_k \lambda_n^k)\right) R^{-1} = R\Big(\text{diag}(f(\lambda_1), \ldots, f(\lambda_n))\Big) R^{-1}.$$

I will briefly deal with how to evaluate $f(A)$ in closed form for non-diagonalizable A on the next homework. The non-diagonalizable case it considerably more involved.

---

5. Compute e^{At} for the matrices A given in exercises 1(c), 2(b) and 2(d). Answer for 2(b):

$$e^{At} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

6. Use the results from the previous exercise to solve the following initial value problems.

(a) $\begin{cases} \dfrac{dx}{dt} = x + 2y, & x(0) = 1, \\ \dfrac{dy}{dt} = -x + 4y, & y(0) = 2. \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = x + y, & x(0) = 2, \\ \dfrac{dy}{dt} = -x + y, & y(0) = 1. \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = x + z, & x(0) = 1, \\ \dfrac{dy}{dt} = y - 2z, & y(0) = 2, \\ \dfrac{dz}{dt} = 2x + 2y + 4z, & z(0) = 3. \end{cases}$

Answer for (b):

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t(2\cos t + \sin t) \\ e^t(\cos t - 2\sin t) \end{pmatrix}.$$

7. For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, determine that $e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

8. Suppose A is a constant square matrix and consider the inhomogeneous IVP

$$\frac{d}{dt}\mathbf{u} + A\mathbf{u} = \mathbf{f}(t), \quad \mathbf{u}(0) = \mathbf{u}_0.$$

(a) Show that the IVP is solved by

$$\mathbf{u}(t) = e^{-At}\mathbf{u}_0 + \int_0^t e^{A(\tau - t)}\mathbf{f}(\tau)\, d\tau.$$

(b) Use part (a) and your result from exercise 7 to find the closed form solution to the coupled inhomogeneous system

$$\begin{cases} \dfrac{dx}{dt} - y = t, & x(0) = 0, \\ \dfrac{dy}{dt} - x = e^t, & y(0) = 0. \end{cases}$$
