This exam has 10 problems, and all 10 problems will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, last name first, and student id number on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. We’ve considered four basic classes of first order ODE’s we can in principle solve. They are: (1) Separable, (2) Linear, (3) Homogeneous, (4) Exact. Classify each of the following but do not solve. An equation may be in more than one class (and you must list all classes if so), or it may be in none. No guessing – show your work!

(a) \( u^2 + 1 + x \frac{du}{dx} = 0 \)  
(b) \( u + 1 + \frac{du}{dx} = 0 \)
(c) \( u + x + x \frac{du}{dx} = 0 \)  
(d) \( u + x + 1 + (u + 2x + 1) \frac{du}{dx} = 0 \)

2. Find the explicit form general solution to each of the following.

(a) \( xe^u \frac{du}{dx} + x + e^u = 0 \)  
(b) \( \frac{du}{dx} = u^2 \)

3. Find the general solution, an implicit form solution is OK, to each of the following.

(a) \( (u + x) \frac{du}{dx} + u + 2x^3 = 0 \)  
(b) \( x^2 \frac{du}{dx} + u^2 + xu = 0 \)

4. A tank is initially filled with 40 gallons of fresh water. A salt brine solution, at two pounds of salt per one gallon of water, is piped into the tank at one gallon per minute. There is another pipe which will output one gallon per minute of the fully mixed water/brine solution. Let \( s(t) \) denote the amount of salt in the tank at time \( t \) measured in minutes.

(a) Set up the first order initial value problem solved by \( s(t) \).

(b) Solve for \( s(t) \).

5. Find the general solution of the following second order differential equations by using the given factorization.

(a) \( \frac{d^2 u}{dx^2} + 2 \frac{du}{dx} + u = \left( \frac{d}{dx} + I \right) \left( \frac{du}{dx} + u \right) = 0 \).

(b) \( \frac{d^2 u}{dx^2} + 3 \frac{du}{dx} + 2u = \left( \frac{d}{dx} + I \right) \left( \frac{du}{dx} + 2u \right) = x \).
6. Determine the general solution to each of the following by using the method of guessing.

(a) \( \frac{d^2 u}{dx^2} - u = e^x \).  
(b) \( \frac{d^2 u}{dx^2} + u = x^2 + 1 \).

7. Solve the following inhomogeneous IVPs using Duhamel’s principle.

(a) \( \frac{d^2 u}{dx^2} + u = 1 \), \( u(0) = u'(0) = 0 \).  
(b) \( \frac{d^2 u}{dx^2} - \frac{du}{dx} = e^x \), \( u(0) = u'(0) = 0 \).

8. Consider the matrix \( A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix} \).

(a) Determine the eigenvalues of \( A \).

(b) Determine all associated eigenvectors.

On part (a), I got \( \lambda = 1, 2, 3 \). You are free to use these on part (b).

9. Use the matrices in this similarity transformation

\[ R^{-1}AR = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \Lambda \]

to do the following.

(a) Compute \( e^{At} \) where \( A = \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix} \).

(b) Solve the initial value problem.

\[ \frac{du}{dt} = -5u + 12v, \quad u(0) = 1, \]
\[ \frac{dv}{dt} = -2u + 5v, \quad v(0) = 0. \]

10. The matrix \( A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \) is not diagonalizable.

(a) Determine \( S \) such that \( S^{-1}AS = J \) where \( J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \).

(b) Compute \( e^{At} \).