First Order ODEs (Part I)

A first order ODE is *separable* when it can be written in the form

$$\frac{du}{dx} = f(x)\,g(u).$$

For example

$$\frac{du}{dx} = \sin(x)\,e^u \quad \text{is separable, whereas} \quad \frac{du}{dx} = \sin(x) + e^u \quad \text{is not.}$$

Here's another example of a separable equation including the solution technique. Find the general solution to

$$\frac{du}{dx} = xu^2.$$

Write the ODE in differential form

$$u^{-2}\,du = x\,dx \quad \Rightarrow \quad \int u^{-2}\,du = \int x\,dx \quad \Rightarrow \quad \frac{u^{-1}}{-1} = c + x^2/2.$$

Note above that <u>everything</u> in the left integral is in terms of u and <u>everything</u> in the right is in terms of x. If this were not so, there would be no way to evaluate the integral. Integration gives u in terms of x defined here via the *implicit* relation $-u^{-1} = c + x^2/2$ where the constant of integration c is arbitrary. An *explicit* solution is one in which u is defined explicitly in terms of x. In the example above we can use the implicit relation

$$-u^{-1} = c + x^2/2 \quad \text{to solve explicily for} \quad u(x) = -\frac{1}{c + x^2/2} = \frac{1}{\tilde{c} - x^2/2}.$$

As an exercise you might want to plug $u(x)$ back into the given differential equation to verify it works.

A first order initial value problem, or IVP for short, imposes that the ODE's solution $u(x)$ takes on a particular value, say u_0, at a given single point, say $x = x_0$. For example, the IVP

$$\frac{du}{dx} = xu^2, \quad u(2) = 3,$$

(here $x_0 = 2$, $u_0 = 3$) can be solved by using the general solution already computed

$$u(x) = \frac{1}{c - x^2/2} \quad \Rightarrow \quad 3 = u(2) = \frac{1}{c - 2^2/2} \quad \Rightarrow \quad c = 7/3.$$

Therefore, the IVP's solution is

$$u(x) = \frac{1}{7/3 - x^2/2}.$$

I think it's a bit easier to just skip the general solution step and compute the IVP's solution for the separable problem by employing definite integrals (as opposed to the indefinite integral I used earlier) as follows

$$u^{-2}\,du = x\,dx \quad \Rightarrow \quad \int_{v=3}^{v=u} v^{-2}\,dv = \int_{y=2}^{y=x} y\,dy \quad \Rightarrow \quad (-v^{-1})\Big|_3^u = (y^2/2)\Big|_2^x \ \ldots$$

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1. Find the general solution (in explicit form) to the following separable first order ODEs.

(a) $\dfrac{du}{dx} = u^2$ (c) $\dfrac{du}{dx} = u^2 + 1$

(b) $\dfrac{du}{dx} = e^{x+u}$ (d) $\dfrac{du}{dx} = (u^2 - u)x$

My answer to (d): $u(x) = 1/(1 + c\,e^{x^2/2})$.

2. Determine the solution to the following IVPs.

(a) $\dfrac{du}{dx} = \sqrt{|u|},\ u(0) = 1.$ (c) $\dfrac{du}{dx} = e^{x-u},\ u(0) = 0.$

(b) $\dfrac{du}{dx} = \sqrt{|u|},\ u(0) = -1.$ (d) $\dfrac{du}{dx} = x(u^2 + 1),\ u(1) = 0.$

Note what happens in part (a) when $x = -2$ and in part (b) when $x = 2$. We'll talk about this phenomenon in some depth later. Also, note what happens in part (d) when $x^2 \to 1 \pm \pi$.

A *first order linear* ODE has the form

$$\frac{du}{dx} + a(x)u = b(x),$$

where $a(x)$ and $b(x)$ are given functions of x. Let $A(x)$ denote the antiderivative of $a(x)$, i.e. $A'(x) = a(x)$, and use the product and chain rule to see that

$$\frac{d}{dx}\left(e^{A(x)}u\right) = e^{A(x)}\left(\frac{du}{dx} + a(x)u\right).$$

Therefore, the first order linear ODE can be rewritten as

$$e^{-A(x)}\frac{d}{dx}\left(e^{A(x)}u\right) = \frac{du}{dx} + a(x)u = b(x) \quad \Rightarrow \quad \frac{d}{dx}\left(e^{A(x)}u\right) = e^{A(x)}b(x).$$

Now, substitute $w(x) = e^{A(x)}u(x)$ in the right above and solve for w in terms of x as you would for any trivially separable equation. Once you have w, the ODE's solution u is given by $u(x) = e^{-A(x)}w(x)$.

Here's an example of a first order linear ODE.

$$\frac{du}{dx} + xu = x^3 \quad \Rightarrow \quad A(x) = \int x\,dx = x^2/2 \quad \Rightarrow \quad \frac{dw}{dx} = x^3 e^{x^2/2}.$$

The integral $\int x^3 e^{x^2/2}\,dx = e^{x^2/2}(x^2 - 2) + c$ is done by substitution, and so we get

$$u(x) = e^{-x^2/2}w(x) = c\,e^{-x^2/2} + x^2 - 2.$$

The IVP can be solved by first finding the general solution (as just done) and then determining what the constant c is. However, here again I prefer to use the definite integral.

For example

$$\frac{du}{dx} + xu = x^3, \quad u(0) = 5 \quad \Rightarrow \quad \int_{y=0}^{y=x} \frac{d}{dy}\left(e^{y^2/2}u(y)\right)\, dy = \int_{y=0}^{y=x} y^3 e^{y^2/2}\, dy,$$

and use $u(0) = 5$ to obtain

$$e^{x^2/2}u(x) - e^0 5 = e^{x^2/2}(x^2 - 2) - e^0(0 - 2) \quad \Rightarrow \quad u(x) = 5e^{-x^2/2} + x^2 - 2 + 2e^{-x^2/2}.$$

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3. Find the general solution to the following linear first order ODEs.

(a) $\dfrac{du}{dx} + u = x$ (c) $\dfrac{du}{dx} = x\,u + x$

(b) $\dfrac{du}{dx} + 2x\,u = e^{-x^2}$ (d) $\dfrac{du}{dx} + \dfrac{1}{x}u = x^3$

Note that part (c) is separable the way it stands. Do this one both ways.

4. Solve the following IVPs.

(a) $\dfrac{du}{dx} = 3u,\ u(0) = 1.$ (c) $\dfrac{du}{dx} - 2u = 1,\ u(0) = 3.$

(b) $\dfrac{du}{dx} + 2x\,u = e^{-x^2},\ u(0) = 2.$ (d) $\dfrac{du}{dx} + \dfrac{1}{x}u = x^3,\ u(1) = 4.$

Here is a mixture of first order IVPs. Each ODE is separable and/or linear.

5. Find the solution to each of the following.

(a) $\dfrac{du}{dx} + u = \dfrac{1}{1+e^x},\ u(0) = 1.$ (d) $x\dfrac{du}{dx} - u = 2x\log x,\ u(e) = 1.$

(b) $\dfrac{du}{dx} = \dfrac{e^{x-u}}{1+e^x},\ u(1) = 0.$ (e) $\dfrac{du}{dx} = (x-1)u^2 - (x-1),\ u(1) = 0.$

(c) $\dfrac{du}{dx} = \dfrac{x}{u},\ u(0) = 1.$ (f) $\dfrac{du}{dx} + \cos x\, u = \sin x \cos x,\ u(0) = 1.$