A first order ODE of the form \( \frac{du}{dx} = f(u/x) \).

is called homogeneous. For example, the ODE
\[
\frac{du}{dx} = \frac{x^2 + u^2}{x^2 + 4u^2}
\]
is homogeneous since
\[
\frac{x^2 + u^2}{x^2 + 4u^2} = \frac{1 + (u/x)^2}{1 + 4(u/x)^2} \equiv f(u/x).
\]

When a first order ODE is homogeneous, it can be made separable by making the substitution \( v = u/x \), giving
\[
\frac{du}{dx} = f(u/x). \quad \Rightarrow \quad \frac{d(xv)}{dx} = x \frac{dv}{dx} + v = f(v) \quad \Rightarrow \quad x \frac{dv}{dx} = f(v) - v.
\]

Consider the example
\[
\frac{du}{dx} - \frac{1}{x} u = 1 \quad \text{or equivalently} \quad \frac{du}{dx} = 1 + u/x.
\]

Written as on the left, recognize this example is first order linear, and it can be solved as such (see your previous homework assignment) yielding \( u = cx + x \log x \). Clearly, this example is also homogeneous, and so set \( v = u/x \)
\[
\frac{du}{dx} = 1 + u/x \quad \Rightarrow \quad x \frac{dv}{dx} + v = 1 + v \quad \Rightarrow \quad dv = \frac{dx}{x} \quad \Rightarrow \quad v = c + \log x.
\]
Therefore, since \( u/x = v = c + \log x \) we get \( u = x(c + \log x) \), in agreement as expected with the solution found using the linear solution technique.

1. The following first order ODEs are homogeneous. Find the general solution. (You may leave your answer in implicit form if convenient.)

(a) \( \frac{du}{dx} = 1 + (u/x) + (u/x)^2 \) \quad (c) \( \frac{du}{dx} = \frac{x-u}{x+u} \)

(b) \( \frac{du}{dx} = \frac{x^2 + u^2}{xu} \) \quad (d) \( \frac{du}{dx} = \frac{u-x}{x-4u} \)

2. Determine the explicit solution to the following IVPs.

(a) \( \frac{du}{dx} = 1 + (u/x) + (u/x)^2 \), \( u(1) = 1 \) \quad (c) \( \frac{du}{dx} = \frac{x-u}{x+u} \), \( u(1) = 0 \).

(b) \( \frac{du}{dx} = \frac{x^2 + u^2}{xu} \), \( u(1) = 2 \) \quad (d) \( \frac{du}{dx} = \frac{x-u}{x} \), \( u(1) = 0 \).

An ODE in the form
\[
a(x, u) \frac{du}{dx} + b(x, u) = 0
\]
is called \textit{exact} when the functions \(a\) and \(b\) satisfy
\[
\frac{\partial}{\partial x} a(x, u) = \frac{\partial}{\partial u} b(x, u).
\]
For example,
\[
2 (x + 1) u \frac{du}{dx} + 3x^2 + u^2 = 0 \text{ is in exact form,}
\]
whereas \(2 (x + 1) u \frac{du}{dx} + 3x^2 + 2u^2 = 0\) is not.

Also, please note the following. Simply scaling an exact ODE will generally yield an ODE that is not exact. For example,
\[
\frac{du}{dx} + 3x^2 + u^2 = 0 \text{ is not in exact form.}
\]

Here’s how to solve an ODE in exact form. Consider the chain rule applied to a two variable function, \(\Lambda(u, x)\), where \(u\) is itself a function of \(x\):
\[
\frac{d}{dx} \Lambda(u, x) = \Lambda_u(u, x) \frac{du}{dx} + \Lambda_x(u, x).
\]
Here I’ve used the very common notation \(\Lambda_u \equiv \partial \Lambda / \partial u\) and \(\Lambda_x \equiv \partial \Lambda / \partial x\). (This form of the chain rule is an important fact seen in Calculus 3.) Suppose we could choose \(\Lambda\) so that
\[
\Lambda_u(u, x) = a(x, u) \quad \text{and} \quad \Lambda_x(u, x) = b(x, u)
\]
\[
\Rightarrow \quad \frac{d}{dx} \Lambda(u, x) = a(x, u) \frac{du}{dx} + b(x, u) = 0 \quad \Rightarrow \quad \Lambda(u, x) = \text{constant}.
\]
The relation \(\Lambda(u, x) = \text{constant}\) now defines \(u(x)\) implicitly. Since generally the second mixed partials satisfy \(\Lambda_{u,x} = \Lambda_{x,u}\), this function \(\Lambda\) can exist (if and) only if \(a_x = b_u\).

Here’s an example calculation to show how to find \(\Lambda\) when an ODE is exact.
\[
2 (x + 1) u \frac{du}{dx} + 3x^2 + u^2 = 0
\]
\[
\Rightarrow \quad \frac{\partial}{\partial u} \Lambda(u, x) = 2 (x + 1) u, \quad \frac{\partial}{\partial x} \Lambda(u, x) = 3x^2 + u^2.
\]
But via (partial) indefinite integration in \(u\), for example, we get
\[
\frac{\partial}{\partial u} \Lambda(u, x) = 2 (x + 1) u \quad \Rightarrow \quad \Lambda(u, x) = \int 2 (x + 1) u \, du = (x + 1) u^2 + h(x).
\]
Here, think of \(x\) as a constant parameter in the \(du\) integral and \(h(x)\) as a constant of integration. Now, differentiate this wrt \(x\)
\[
\frac{\partial}{\partial x} \Lambda(u, x) = u^2 + h'(x) \quad \text{but using} \quad \frac{\partial}{\partial x} \Lambda(u, x) = 3x^2 + u^2 \quad \Rightarrow \quad h'(x) = 3x^2.
\]
Therefore \(h(x) = x^3\), and we finally get
\[
\Lambda(u, x) = (x + 1) u^2 + x^3 = c \quad \Rightarrow \quad u = \pm \sqrt{\frac{c - x^3}{x + 1}}.
\]
BTW. I could have just as well started with $\Lambda$ in the previous example. Let me show you.

$$\frac{\partial}{\partial x} \Lambda(u, x) = 3x^2 + u^2 \quad \Rightarrow \quad \Lambda(u, x) = \int (3x^2 + u^2) \, dx = x^3 + u^2 x + g(u).$$

Differentiate wrt $u$

$$\frac{\partial}{\partial u} \Lambda(u, x) = 2ux + g'(u) \quad \text{but using} \quad \frac{\partial}{\partial u} \Lambda(u, x) = 2(x+1)u \quad \Rightarrow \quad g'(u) = 2u.$$

Therefore $g(u) = u^2$ and so

$$\Lambda(u, x) = x^3 + u^2 x + u^2 = (x+1)u^2 + x^3$$

just like obtained before.

3. Find the general solution to each of the following exact ODEs. Try to solve for $u(x)$ explicitly.

(a) $(u + x) \frac{du}{dx} + u + 2x^3 = 0$
(b) $xe^u \frac{du}{dx} + e^u + 1 = 0$
(c) $(2xu + 2) \frac{du}{dx} + u^2 + 4x^3 = 0$
(d) $(u^3 - x^2u) \frac{du}{dx} - xu^2 = 0$

4. So far, we’ve considered four basic classes of first order ODEs we can in principle solve. They are: (1) Separable, (2) Linear, (3) Homogeneous, (4) Exact. Classify each of the following but do not solve. Please note: An equation may be in more than one class, or it may be in none.

(a) $x \frac{du}{dx} + u + x = 0$
(b) $u \frac{du}{dx} + u + x = 0$
(c) $(x^2 + 1) \frac{du}{dx} + u^2 = 0$
(d) $(u^2 + 1) \frac{du}{dx} + e^{x+u} = 0$
(e) $(u^2 + x^2) \frac{du}{dx} + e^u = 0$
(f) $(u + x + 2) \frac{du}{dx} + (u + x + 1) = 0$
(g) $(x^2 + 1) \frac{du}{dx} + xu + x^2 = 0$
(h) $(u + x) \frac{du}{dx} + 2u + 2 = 0$

Almost every technique employed to solve an ODE involves some kind of clever change of variables. The exact example I discussed earlier

$$2(x+1)u \frac{du}{dx} + 3x^2 + u^2 = 0$$

can be rewritten as

$$\frac{dv}{dx} + \frac{1}{x+1}v = -\frac{3x^2}{x+1}$$

by using the fact that $2u \frac{du}{dx} = \frac{du^2}{dx}$ and substituting $v = u^2$. The resulting first order linear equation can be solved just as you did on your previous homework assignment yielding $(x+1)v = c - x^3$. Of course this agrees with the solution we got above by using the exact equation trick.
Recall earlier I used
\[ 2(x + 1)u \frac{du}{dx} + 3x^2 + 2u^2 = 0 \]
to illustrate an ODE that is not exact. In fact, it’s not separable, it’s not linear nor is it homogeneous. However, again substituting \( v = u^2 \) casts this equation into first order linear form
\[ \frac{dv}{dx} + \frac{2}{x+1}v = -\frac{3x^2}{x+1} \]
which you can easily solve.

Here’s one more substitution example. An ODE of the form
\[ \frac{du}{dx} + a(x)u = b(x)u^p \text{ with } p \neq 0, 1 \]
is called a Bernoulli equation. Substitute \( u = v^\gamma \) (the constant \( \gamma \) will be determined) to find
\[
\begin{align*}
\frac{dv^\gamma}{dx} + a(x)v^\gamma &= b(x)v^{\gamma p} \\
\Rightarrow \quad \gamma v^{\gamma-1} \frac{dv}{dx} + a(x)v^\gamma &= b(x)v^{\gamma p} \\
\Rightarrow \quad \gamma \frac{dv}{dx} + a(x)v &= b(x)v^{\gamma p+1-\gamma}.
\end{align*}
\]
Of course I’m assuming \( v \neq 0 \) to justify the division in the last step. Now, take \( \gamma \) so that \( \gamma p + 1 - \gamma = 0 \), that is \( \gamma = 1/(1-p) \). This yields
\[
\frac{dv}{dx} + (1-p)a(x)v = (1-p)b(x).
\]
But this is a first order linear problem in \( v \), and in principle we know how to solve it.

5. Find the general solution to each of the following Bernoulli equations.

(a) \( \frac{du}{dx} + u = u^4 \)  \hspace{1cm} (b) \( u^2 + xu + x^2 \frac{du}{dx} = 0 \)

Don’t try to separate (a) – that’ll be way too hard.

6. The ODE in exercise 4(h)
\[ (u + x) \frac{du}{dx} + 2u + 2 = 0 \]
doesn’t seem to fit into any of the categories we’ve so far discussed.

(a) Change dependent and independent variables, \( v = u + 1 \) and \( y = x - 1 \), to cast this equation into
\[ (v + y) \frac{dv}{dy} + 2v = 0. \]

(b) Solve this homogeneous ODE for \( v(y) \), an implicit relation between \( v \) and \( y \) will do, and then change variables back to find an implicit relation satisfied by \( u(x) \). Answer: I got
\[ (u + 1)(u - 2 + 3x)^2 = c. \]