First Order ODEs (Part II)

A first order ODE of the form

$$\frac{du}{dx} = f(u/x).$$

is called *homogeneous*. For example, the ODE

$$\frac{du}{dx} = \frac{x^2+u^2}{x^2+4u^2} \quad \text{is homogeneous since} \quad \frac{x^2+u^2}{x^2+4u^2} = \frac{1+(u/x)^2}{1+4(u/x)^2} \equiv f(u/x).$$

When a first order ODE is homogeneous, it can be made separable by making the substitution $v = u/x$, giving

$$\frac{du}{dx} = f(u/x). \quad \Rightarrow \quad \frac{d(xv)}{dx} = x\frac{dv}{dx} + v = f(v) \quad \Rightarrow \quad x\frac{dv}{dx} = f(v) - v.$$

Consider the example

$$\frac{du}{dx} - \frac{1}{x}u = 1 \;\; \text{or equivalently} \;\; \frac{du}{dx} = 1 + u/x.$$

Written as on the left, recognize this example is first order linear, and it can be solved as such (see your previous homework assignment) yielding $u = cx + x\log x$. Clearly, this example is also homogeneous, and so set $v = u/x$

$$\frac{du}{dx} = 1 + u/x \quad \Rightarrow \quad x\frac{dv}{dx} + v = 1 + v \quad \Rightarrow \quad dv = \frac{dx}{x} \quad \Rightarrow \quad v = c + \log x.$$

Therefore, since $u/x = v = c + \log x$ we get $u = x(c + \log x)$, in agreement as expected with the solution found using the linear solution technique.

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1. The following first order ODEs are homogeneous. Find the general solution. (You may leave your answer in <u>implicit</u> form if convenient.)

(a) $\dfrac{du}{dx} = 1 + (u/x) + (u/x)^2$ (c) $\dfrac{du}{dx} = \dfrac{x-u}{x+u}$

(b) $\dfrac{du}{dx} = \dfrac{x^2+u^2}{xu}$ (d) $\dfrac{du}{dx} = \dfrac{u-x}{x-4u}$

2. Determine the <u>explicit</u> solution to the following IVPs.

(a) $\dfrac{du}{dx} = 1 + (u/x) + (u/x)^2,\ u(1) = 1.$ (c) $\dfrac{du}{dx} = \dfrac{x-u}{x+u},\ u(1) = 0.$

(b) $\dfrac{du}{dx} = \dfrac{x^2+u^2}{xu},\ u(1) = 2.$ (d) $\dfrac{du}{dx} = \dfrac{x-u}{x},\ u(1) = 0.$

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An ODE in the form

$$a(x,u)\,\frac{du}{dx} + b(x,u) = 0$$

is called *exact* when the functions $a$ and $b$ satisfy

$$\frac{\partial}{\partial x} a(x,u) = \frac{\partial}{\partial u} b(x,u).$$

For example,

$$2\,(x+1)\,u\,\frac{du}{dx} + 3x^2 + u^2 = 0 \;\text{ is in exact form,}$$
$$\text{whereas }\; 2\,(x+1)\,u\,\frac{du}{dx} + 3x^2 + 2u^2 = 0 \;\text{ is not.}$$

Also, please note the following. Simply scaling an exact ODE will generally yield an ODE that is not exact. For example,

$$\frac{du}{dx} + \frac{3x^2+u^2}{2\,(x+1)\,u} = 0 \;\text{ is not in exact form.}$$

Here's how to solve an ODE in exact form. Consider the chain rule applied to a two variable function, $\Lambda(u,x)$, where $u$ is itself a function of $x$:

$$\frac{d}{dx}\Lambda(u,x) = \Lambda_u(u,x)\,\frac{du}{dx} + \Lambda_x(u,x).$$

Here I've used the very common notation $\Lambda_u \equiv \partial\Lambda/\partial u$ and $\Lambda_x \equiv \partial\Lambda/\partial x$. (This form of the chain rule is an important fact seen in Calculus 3.) Suppose we could choose $\Lambda$ so that

$$\Lambda_u(u,x) = a(x,u) \;\text{ and }\; \Lambda_x(u,x) = b(x,u)$$
$$\Rightarrow\quad \frac{d}{dx}\Lambda(u,x) = a(x,u)\,\frac{du}{dx} + b(x,u) = 0 \quad\Rightarrow\quad \Lambda(u,x) = \textit{constant}.$$

The relation $\Lambda(u,x) = \textit{constant}$ now defines $u(x)$ implicitly. Since generally the second mixed partials satisfy $\Lambda_{u,x} = \Lambda_{x,u}$, this function $\Lambda$ can exist (if and) only if $a_x = b_u$.

Here's an example calculation to show how to find $\Lambda$ when an ODE is exact.

$$2\,(x+1)\,u\,\frac{du}{dx} + 3x^2 + u^2 = 0$$
$$\Rightarrow\quad \frac{\partial}{\partial u}\Lambda(u,x) = 2\,(x+1)\,u, \quad \frac{\partial}{\partial x}\Lambda(u,x) = 3x^2 + u^2.$$

But via (partial) indefinite integration in $u$, for example, we get

$$\frac{\partial}{\partial u}\Lambda(u,x) = 2\,(x+1)\,u \quad\Rightarrow\quad \Lambda(u,x) = \int 2\,(x+1)\,u\,du = (x+1)\,u^2 + h(x).$$

Here, think of $x$ as a constant parameter in the $du$ integral and $h(x)$ as a constant of integration. Now, differentiate this wrt $x$

$$\frac{\partial}{\partial x}\Lambda(u,x) = u^2 + h'(x) \;\text{ but using }\; \frac{\partial}{\partial x}\Lambda(u,x) = 3x^2 + u^2 \quad\Rightarrow\quad h'(x) = 3x^2.$$

Therefore $h(x) = x^3$, and we finally get

$$\Lambda(u,x) = (x+1)\,u^2 + x^3 = c \quad\Rightarrow\quad u = \pm\sqrt{\frac{c-x^3}{x+1}}.$$

BTW. I could have just as well started with $\Lambda_x$ in the previous example. Let me show you.

$$\frac{\partial}{\partial x}\Lambda(u,x) = 3x^2 + u^2 \quad \Rightarrow \quad \Lambda(u,x) = \int (3x^2+u^2)\,dx = x^3 + u^2 x + g(u).$$

Differentiate wrt $u$

$$\frac{\partial}{\partial u}\Lambda(u,x) = 2ux + g'(u) \text{ but using } \frac{\partial}{\partial u}\Lambda(u,x) = 2(x+1)u \quad \Rightarrow \quad g'(u) = 2u.$$

Therefore $g(u) = u^2$ and so

$$\Lambda(u,x) = x^3 + u^2 x + u^2 = (x+1)\,u^2 + x^3$$

just like obtained before.

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3. Find the general solution to each of the following exact ODEs. Try to solve for $u(x)$ explicitly.

(a) $(u+x)\,\dfrac{du}{dx} + u + 2x^3 = 0$ (c) $(2xu+2)\,\dfrac{du}{dx} + u^2 + 4x^3 = 0$

(b) $xe^u\,\dfrac{du}{dx} + e^u + 1 = 0$ (d) $(u^3 - x^2 u)\,\dfrac{du}{dx} - xu^2 = 0$

4. So far, we've considered four basic classes of first order ODEs we can in principle solve. They are: (1) Separable, (2) Linear, (3) Homogeneous, (4) Exact. Classify each of the following but do not solve. Please note: An equation may be in more than one class, or it may be in none.

(a) $x\,\dfrac{du}{dx} + u + x = 0$ (e) $(u^2+x^2)\,\dfrac{du}{dx} + e^u = 0$

(b) $u\,\dfrac{du}{dx} + u + x = 0$ (f) $(u+x+2)\,\dfrac{du}{dx} + (u+x+1) = 0$

(c) $(x^2+1)\,\dfrac{du}{dx} + u^2 = 0$ (g) $(x^2+1)\,\dfrac{du}{dx} + xu + x^2 = 0$

(d) $(u^2+1)\,\dfrac{du}{dx} + e^{x+u} = 0$ (h) $(u+x)\,\dfrac{du}{dx} + 2u + 2 = 0$

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Almost every technique employed to solve an ODE involves some kind of clever change of variables. The exact example I discussed earlier

$$2\,(x+1)\,u\,\frac{du}{dx} + 3x^2 + u^2 = 0 \text{ can be rewritten as } \frac{dv}{dx} + \frac{1}{x+1}v = -\frac{3x^2}{x+1}$$

by using the fact that $2u\,du/dx = du^2/dx$ and substituting $v = u^2$. The resulting first order linear equation can be solved just as you did on your previous homework assignment yielding $(x+1)v = c - x^3$. Of course this agrees with the solution we got above by using the exact equation trick.

Recall earlier I used

$$2\,(x+1)\,u\,\frac{du}{dx}+3x^2+2u^2=0$$

to illustrate an ODE that is not exact. In fact, it's not separable, it's not linear nor is it homogeneous. However, again substituting $v=u^2$ casts this equation into first order linear form

$$\frac{dv}{dx}+\frac{2}{x+1}v=-\frac{3x^2}{x+1}$$

which you can easily solve.

Here's one more substitution example. An ODE of the form

$$\frac{du}{dx}+a(x)u=b(x)u^p \text{ with } p\neq 0,1$$

is called a *Bernoulli equation.* Substitute $u=v^\gamma$ (the constant $\gamma$ will be determined) to find

$$\begin{aligned}
&\frac{dv^\gamma}{dx}+a(x)v^\gamma=b(x)v^{\gamma p}\\
\Rightarrow\quad&\gamma v^{\gamma-1}\frac{dv}{dx}+a(x)v^\gamma=b(x)v^{\gamma p}\\
\Rightarrow\quad&\gamma\frac{dv}{dx}+a(x)v=b(x)v^{\gamma p+1-\gamma}.
\end{aligned}$$

Of course I'm assuming $v\neq 0$ to justify the division in the last step. Now, take $\gamma$ so that $\gamma p+1-\gamma=0$, that is $\gamma=1/(1-p)$. This yields

$$\frac{dv}{dx}+(1-p)\,a(x)v=(1-p)\,b(x).$$

But this is a first order linear problem in $v$, and in principle we know how to solve it.

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5. Find the general solution to each of the following Bernoulli equations.

(a) $\dfrac{du}{dx}+u=u^4$ (b) $u^2+xu+x^2\,\dfrac{du}{dx}=0$

Don't try to separate (a) – that'll be way too hard.

6. The ODE in exercise 4(h)

$$(u+x)\,\frac{du}{dx}+2u+2=0$$

doesn't seem to fit into any of the categories we've so far discussed.

(a) Change dependent and independent variables, $v=u+1$ and y $=x-1$, to cast this equation into

$$(v+y)\,\frac{dv}{dy}+2v=0.$$

(b) Solve this homogeneous ODE for $v(y)$, an implicit relation between $v$ and $y$ will do, and then change variables back to find an implicit relation satisfied by $u(x)$. Answer: I got $(u+1)(u-2+3x)^2=c$.

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