Grönwall and First Order Systems

Suppose $w(t)$ is a real valued and continuous function on an open interval $-\delta < t-t_0 < \delta$. Also, let $w_0$ and $L$ denote nonnegative constants. One form of the celebrated *Grönwall inequality* can be stated as follows.

$$\text{Given that for all such } t \;\; w(t) \le w_0 + L\int_{\min(t,t_0)}^{\max(t,t_0)} w(\tau)\,d\tau \quad \Rightarrow \quad w(t) \le e^{L|t-t_0|}w_0.$$

Before showing you applications of Grönwall, let's first derive it. Clearly the result is obvious when $L=0$, so let's work on the case when $L>0$. Also, let's assume for now that $0 \le t-t_0 < \delta$. Define

$$v(t) \equiv \int_{t_0}^{t} w(\tau)\,d\tau \quad \Rightarrow \quad \frac{d}{dt}v(t) = w(t) \quad \Rightarrow \quad \frac{dv}{dt} \le w_0 + Lv.$$

Just as we did the first week of class, see that

$$\begin{aligned}
&\frac{dv}{dt} - Lv \le w_0 \quad \Rightarrow \quad e^{Lt}\frac{d}{dt}\left(e^{-Lt}v(t)\right) \le w_0 \\
\Rightarrow \quad & \int_{t_0}^{t} \frac{d}{d\tau}\left(e^{-L\tau}v(\tau)\right)\,d\tau \le \int_{t_0}^{t} e^{-L\tau}w_0\,d\tau \\
\Rightarrow \quad & e^{-Lt}v(t) - e^{-Lt_0}v(t_0) \le -\frac{1}{L}\left(e^{-Lt} - e^{-Lt_0}\right)w_0 \\
\Rightarrow \quad & v(t) \le \frac{1}{L}\left(e^{L(t-t_0)} - 1\right)w_0.
\end{aligned}$$

Notice above I used $v(t_0)=0$. Therefore, for $0 \le t-t_0 < \delta$, we have

$$w(t) \le w_0 + L\int_{t_0}^{t} w(\tau)\,d\tau \quad \Rightarrow \quad w(t) \le w_0 + L\left(\frac{1}{L}\left(e^{L(t-t_0)} - 1\right)w_0\right) = e^{L(t-t_0)}w_0.$$

Next, when $t$ is in the range $-\delta < t-t_0 \le 0$, a similar argument as given above will also show $w(t) \le e^{L(t_0-t)}w_0$. So, for any $-\delta < t-t_0 < \delta$, we've established the fact that $w(t) \le e^{L|t-t_0|}w_0$ which is what the Grönwall inequality says.

As you know, solutions to IVPs can *blow-up* in finite time. For example, consider the first order scalar IVP

$$\frac{du}{dt} = u^2 \;\;\text{with}\;\; u(0)=1.$$

Solve this simple separable equation to get $u(t) = 1/(1-t)$. However, observe that as $t$ approaches $t=1$ from below, the solution blows-up, i.e. $\lim_{t\uparrow 1} u(t) = \infty$. On the other hand, the solution to

$$\frac{du}{dt} = u \;\;\text{with}\;\; u(0)=1$$

is $u(t)=e^t$, and it remains well defined for any $t$.

Here's a little theorem about global existence of solutions to IVPs. Consider

$$\frac{du}{dt} = f(u,t) \quad \text{with} \quad u(0) = u_0,$$

where $f(u,t)$ is a continuous function for all $u$ and $t$. Suppose in addition $f$ satisfies the growth condition

$$\max_{|t|\le T} |f(u,t)| \le a(T)|u| + b(T)$$

for any finite $T$. Then the IVP's solution, $u(t)$, exits for all $t$, i.e. $u(t)$ does <u>not</u> blow-up in finite time. First, let me state without proof the following local existence result. The continuity assumption concerning $f$ implies there is an open interval $-\delta < t < \delta$ on which the IVP has a continuously differentiable solution $u(t)$. Moreover, this interval can be extended to all $t$ provided $|u(t)|$ remains bounded on any finite $t$ interval. Let's use Grönwall together with the given growth condition to show this is true.

Integrate the differential equation from $0$ to $t$ with $|t| \le T$ and use the triangle inequality to see

$$\begin{aligned} u(t) - u(0) &= \int_0^t \frac{du}{d\tau}\, d\tau = \int_0^t f(u(\tau),\tau)\, d\tau \\ \Rightarrow \quad |u(t)| &\le |u_0| + \int_{\min(t,0)}^{\max(t,0)} |f(u(\tau),\tau)|\, d\tau. \end{aligned}$$

The growth condition gives

$$\begin{aligned} |u(t)| &\le |u_0| + \int_{\min(t,0)}^{\max(t,0)} \left(a(T)|u(\tau)| + b(T)\right)\, d\tau \\ &\le (|u_0| + b(T)\,T) + a(T) \int_{\min(t,0)}^{\max(t,0)} |u(\tau)|\, d\tau, \end{aligned}$$

and with this, Grönwall yields

$$|u(t)| \le (|u_0| + b(T)\,T)\, e^{a(T)|t-0|} \quad \text{whenever } |t| \le T.$$

Therefore, we conclude that $|u(t)|$ remains bounded on any finite $t$ interval.

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Conclude the solution to each of the following scalar first order IVPs remains bounded on any finite $t$ interval. Don't try to solve.

1. $\dfrac{du}{dt} = \dfrac{u\sin(u)}{u^2+1}$, with $u(0) = 1$.
2. $\dfrac{du}{dt} = (tu+1)e^{-u^2}$, with $u(0) = 2$.
3. $\dfrac{du}{dt} = \log(t^2u^2+1)$, with $u(0) = 3$.
4. $\dfrac{du}{dt} = \dfrac{u^3+t^3}{u^2+1}$, with $u(0) = 4$.

Hints: You are free to use the following estimates in order to apply the "growth condition" given above.

1. $\left|\dfrac{u\sin(u)}{u^2+1}\right| \le \dfrac{1}{2}$.

2. $\left|(tu+1)e^{-u^2}\right| \le |t|\dfrac{1}{\sqrt{2}}e^{-1/2}+1$.

3. $\left|\log(t^2u^2+1)\right| \le \dfrac{1}{e}\left(|t||u|+1\right)$

4. $\left|\dfrac{u^3+t^3}{u^2+1}\right| \le |u|+|t|^3$.

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Second order IVPs can be written as first order systems. Let me demonstrate. Consider the general second order linear problem

$$\frac{d^2u}{dt^2}+a(t)\frac{du}{dt}+b(t)u = f(t)$$
$$u(t_0)=u_0,\ u_t(t_0)=u_1.$$

Let $v = du/dt$, and so $dv/dt = d^2u/dt^2$, to write

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -a(t)v-b(t)u+f(t) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

This can be symbolically written as

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u},t), \quad \mathbf{u}(t_0) = \mathbf{u}_0,$$

where here I've used bold face letters to indicate vector quantities.

In fact, any order IVP can be written as a first order system. Here's a third order nonlinear example.

$$\frac{d^3u}{dt^3}+\sin(u)\frac{du}{dt} = 0$$
$$u(0)=1,\ u_t(0)=2,\ u_{tt}(0)=3.$$

Let $v = du/dt$ and $w = dv/dt$ and so $dw/dt = d^3u/dt^3$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\sin(u)v \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

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Write each of the following scalar IVPs as a first order system.

5. $\dfrac{d^2u}{dt^2}+\dfrac{2}{t}\dfrac{du}{dt}+\dfrac{1}{t^2}u = e^t, \quad u(1)=1,\ u_t(1)=2.$

6. $\dfrac{d^2u}{dt^2}-t^2\dfrac{du}{dt}+t^4u+\cos(t) = 0, \quad u(0)=2,\ u_t(0)=3.$

7. $\dfrac{d^3u}{dt^3}-2\dfrac{d^2u}{dt^2}-3\dfrac{du}{dt}-4u = 0, \quad u(0)=1,\ u_t(0)=2,\ u_{tt}(0)=3.$

8. $\dfrac{d^4u}{dt^4} = u\dfrac{d^2u}{dt^2}, \quad u(5)=1,\ u_t(5)=u_{tt}(5)=u_{ttt}(5)=0.$

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You all know what a real vector space is. A *norm* is a nonnegative real valued function which acts on any vector $\mathbf{u}$ in the vector space and is denoted by $||\mathbf{u}||$. In addition to $||\mathbf{u}|| \geq 0$, a norm must satisfy each of the following properties.

(1) $||\mathbf{u}|| = 0 \;\Leftrightarrow\; \mathbf{u} = \mathbf{0}$.

(2) $||\alpha \mathbf{u}|| = |\alpha| ||\mathbf{u}||$ for any real scalar $\alpha$.

(3) $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

You saw the Euclidean norm in calculus. For example, on $\mathbb{R}^2$, given a vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{its Euclidean norm is given by} \quad ||\mathbf{u}|| \equiv \sqrt{(u_1)^2 + (u_2)^2}.$$

However, there are an innumerable number of different norms defined on $\mathbb{R}^n$. The one I'll use here is often called the *one–norm*. On $\mathbb{R}^n$, the one–norm of $\mathbf{u}$ is given by

$$||\mathbf{u}|| \equiv \sum_{i=1}^{n} |u_i| = |u_1| + \cdots + |u_n|.$$

You should check that the one–norm does in fact satisfy the properties required of a norm that I listed above.

Now, consider the first order system

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t) \quad \text{with} \quad \mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{f}(\mathbf{u}, t)$ is a continuous function of $\mathbf{u}$ and $t$, and suppose $\mathbf{f}$ satisfies the vector analogue of the growth condition given earlier

$$\max_{|t|\leq T} ||\mathbf{f}(\mathbf{u}, t)|| \leq a(T)||\mathbf{u}|| + b(T) \quad \text{for any finite } T.$$

Then, in a manner essentially identical to what I did for the scalar problem, one can show this system's solution, $\mathbf{u}(t)$, exits for all $t$. In particular, the solution satisfies

$$||\mathbf{u}(t)|| \leq (||\mathbf{u}_0|| + b(T)\,T)\, e^{a(T)|t-0|} \quad \text{for any } |t| \leq T \text{ where } T > 0 \text{ is arbitrary.}$$

Let's apply this result to the third order example

$$\begin{gathered} \frac{d^3u}{dt^3} + \sin(u)\frac{du}{dt} = 0 \\ u(0) = 1, \;\; u_t(0) = 2, \;\; u_{tt}(0) = 3, \end{gathered}$$

which we wrote earlier as the following first order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\sin(u)v \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

It's easy to see for this example that

$$\left|\left| \begin{pmatrix} v \\ w \\ -\sin(u)v \end{pmatrix} \right|\right| = |v| + |w| + |\sin(u)v| \leq 2|v| + |w| \;\; \leq 2 \left|\left| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right|\right|.$$

So, take $a(T) = 2$ and $b(T) = 0$ in the growth condition, and conclude $||\mathbf{u}(t)|| \leq ||\mathbf{u}_0||\, e^{2|t|}$. Also, since $||\mathbf{u}|| = |u| + |v| + |w|$ and $v = du/dt$ and $w = d^2u/dt^2$, we can write this inequality as

$$|u(t)| + \left|\frac{du(t)}{dt}\right| + \left|\frac{d^2u(t)}{dt^2}\right| \leq 6\, e^{2|t|},$$

where this solution estimate is valid for any finite $t$.

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9. Consider the general second order, linear but variable coefficient, scalar IVP

$$\begin{aligned} &\frac{d^2u}{dt^2} + a(t)\frac{du}{dt} + b(t)u = f(t) \\ &u(t_0) = u_0,\ u_t(t_0) = u_1, \end{aligned}$$

where $a(t)$, $b(t)$ and $f(t)$ are continuous functions of $t$. You may assume there is a bounded and open $t$ interval, say $\{t : |t - t_0| < T\}$ for some fixed $0 < T < \infty$, on which the given IVP has a twice continuously differential solution. Show that $|u| + |u_t|$ is bounded on any such interval.

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While the solution to a nonlinear IVP can blow-up in finite time, it does not do so instantaneously. In fact, the solution stays close to its initial datum for a measurable amount of time. To see this, consider the system

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(t_0) = \mathbf{u}_0,$$

where the right hand side $\mathbf{f}(\mathbf{u}, t)$ is continuous at $\mathbf{u}_0$, $t_0$. Recall what continuity says here: For any $\epsilon > 0$, there are numbers $\delta_{\mathbf{u}} > 0$ and $\delta_t > 0$ such that

$$||\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{u}_0, t_0)|| < \epsilon \ \text{ whenever } \ ||\mathbf{u} - \mathbf{u}_0|| < \delta_{\mathbf{u}} \text{ and } |t - t_0| < \delta_t.$$

In particular, take $\epsilon = 1$ to conclude

$$\begin{aligned} & ||\mathbf{f}(\mathbf{u}, t)|| - ||\mathbf{f}(\mathbf{u}_0, t_0)|| \leq ||\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{u}_0, t_0)|| < 1 \\ & \text{for all } \mathbf{u} \text{ and } t \text{ satisfying } \ ||\mathbf{u} - \mathbf{u}_0|| < \delta_{\mathbf{u}},\ |t - t_0| < \delta_t, \\ \Rightarrow \quad & ||\mathbf{f}(\mathbf{u}, t)|| < 1 + ||\mathbf{f}(\mathbf{u}_0, t_0)||. \end{aligned}$$

(I've used the so-called backward triangle inequality on the first line on the left.) Therefore, for any $t$ satisfying

$$|t - t_0| < \delta \equiv \min(\delta_{\mathbf{u}}/(1 + ||\mathbf{f}(\mathbf{u}_0, t_0)||), \delta_t),$$

integrate the differential equation and use continuity to see

$$\mathbf{u}(t) - \mathbf{u}_0 = \int_{t_0}^{t} \mathbf{f}(\mathbf{u}(\tau), \tau)\, d\tau \quad \Rightarrow \quad ||\mathbf{u}(t) - \mathbf{u}_0|| \leq (1 + ||\mathbf{f}(\mathbf{u}_0, t_0)||)\, |t - t_0| < \delta_{\mathbf{u}}.$$

That is,

$$\text{for any } t_0 - \delta < t < t_0 + \delta \quad \Rightarrow \quad ||\mathbf{u}(t) - \mathbf{u}_0|| \leq (1 + ||\mathbf{f}(\mathbf{u}_0, t_0)||)\,|t - t_0|.$$

I want to finish by addressing the question of uniqueness. Continuity of $\mathbf{f}$ gives (at least local) existence of solutions to an IVP, but in general we need a bit more to guarantee the solution is unique. Let's consider one more example.

$$\frac{du}{dt} = \sqrt{|u|} \ \text{ with } \ u(0) = 0.$$

Clearly, one solution to this IVP is $u(t) = 0$. However, you can separate variables to also get

$$u(t) = \begin{cases} t^2/4 & \text{if } t \geq 0 \\ -t^2/4 & \text{if } t < 0. \end{cases}$$

In fact, there are actually four continuously differentiable solutions to this IVP. They are:

$$u_1(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases} \qquad u_2(t) = \begin{cases} t^2/4 & \text{if } t \geq 0 \\ -t^2/4 & \text{if } t < 0, \end{cases}$$

$$u_3(t) = \begin{cases} t^2/4 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases} \qquad u_4(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ -t^2/4 & \text{if } t < 0. \end{cases}$$

Now, $\sqrt{|u|}$ is continuous everywhere and in particular in a neighborhood of the initial datum, $u(0) = 0$. However, it fails to satisfy a somewhat stronger condition we'll need to guarantee an IVP has a unique solution.

Let me now state the fundamental uniqueness theorem. Consider

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(t_0) = \mathbf{u}_0,$$

where the right hand side $\mathbf{f}(\mathbf{u}, t)$, beyond being continuous in $\mathbf{u}$ and $t$, also satisfies

$$||\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{v}, t)|| \leq L||\mathbf{u} - \mathbf{v}||$$

for all $\mathbf{u}$ and $\mathbf{v}$ in a neighborhood of $\mathbf{u}_0$, say $B(\mathbf{u}_0, \delta_{\mathbf{u}})$, and all $t$ in a neighborhood of $t_0$, say $B(t_0, \delta_t)$. That is, we assume $\mathbf{f}(\mathbf{u}, t)$ is what is commonly known as *Lipschitz continuous* in a neighborhood of $\mathbf{u}_0, t_0$. Then (at least locally) the IVP's solution is unique.

Here's the proof. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two solutions of

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(t_0) = \mathbf{u}_0, \qquad \frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v}, t), \quad \mathbf{v}(t_0) = \mathbf{u}_0.$$

Integrate

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{u}(\tau), \tau)\, d\tau, \quad \mathbf{v}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{v}(\tau), \tau)\, d\tau,$$

subtract

$$\mathbf{u}(t) - \mathbf{v}(t) = \mathbf{u}_0 - \mathbf{u}_0 + \int_{t_0}^{t} \left(\mathbf{f}(\mathbf{u}(\tau), \tau) - \mathbf{f}(\mathbf{v}(\tau), \tau)\right)\, d\tau,$$

and apply the triangle inequality

$$||\mathbf{u}(t) - \mathbf{v}(t)|| \le \int_{\min(t,t_0)}^{\max(t,t_0)} ||\mathbf{f}(\mathbf{u}(\tau),\tau) - \mathbf{f}(\mathbf{v}(\tau),\tau)||\, d\tau.$$

At the top of the previous page we showed there is a $\delta > 0$ such that for all $t_0 - \delta < t < t_0 + \delta$

$$||\mathbf{u}(t) - \mathbf{u}_0|| < \delta_{\mathbf{u}}, \quad ||\mathbf{v}(t) - \mathbf{u}_0|| < \delta_{\mathbf{u}}, \quad \Rightarrow \quad \mathbf{u}(t),\ \mathbf{v}(t) \in B(\mathbf{u}_0, \delta_{\mathbf{u}}).$$

Therefore, since $\mathbf{f}(\mathbf{u}, t)$ is assumed to be Lipschitz in this neighborhood,

$$||\mathbf{u}(t) - \mathbf{v}(t)|| \le \int_{\min(t,t_0)}^{\max(t,t_0)} ||\mathbf{f}(\mathbf{u}(\tau),\tau) - \mathbf{f}(\mathbf{v}(\tau),\tau)||\, d\tau \le \int_{\min(t,t_0)}^{\max(t,t_0)} L\,||\mathbf{u}(\tau) - \mathbf{v}(\tau)||\, d\tau,$$

or in other words, for any $t_0 - \delta < t < t_0 + \delta$

$$||\mathbf{u}(t) - \mathbf{v}(t)|| \le L \int_{\min(t,t_0)}^{\max(t,t_0)} ||\mathbf{u}(\tau) - \mathbf{v}(\tau)||\, d\tau.$$

From this, Grönwall tells us the following.

For all $t_0 - \delta < t < t_0 + \delta$ we have $||\mathbf{u}(t) - \mathbf{v}(t)|| \le e^{L|t-t_0|} 0 = 0,$

which is equivalent to saying

$\mathbf{u}(t) = \mathbf{v}(t)$ for every $t$ in the range $t_0 - \delta < t < t_0 + \delta$.

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Solve the given IVPs but only locally in a neighborhood of $t = 0$. Check whether or not the right hand side is Lipschitz around the initial point, i.e. $u(0)$, and compare your results with the statement of the uniqueness theorem.

10. Find all solutions of $\dfrac{du}{dt} = \sqrt[3]{u}$ with the given initial datum.

(a) $u(0) = 0$. (b) $u(0) = 1$.

For (a) I got $u(t) = 0$ as well as $u(t) = \left(\frac{2}{3}t\right)^{3/2}$ for $t \ge 0$. For (b) $u(t) = \left(1 + \frac{2}{3}t\right)^{3/2}$.

11. Find all solutions of $\dfrac{du}{dt} = \sqrt{|1 - u^2|}$ with the given initial datum.

(a) $u(0) = 1$. (b) $u(0) = 0$.

For (a) I got $u(t) = 1$ as well as $u(t) = \cos(t)$ when $t \le 0$ and $u(t) = \cosh(t)$ when $t \ge 0$. Use these to get four separate solutions. For (b) $u(t) = \sin(t)$ is the unique solution.

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