Introduction

An operator $\mathcal{L}(u)$ from one function space to another is said to be *linear* if for all functions u_1 and u_2 in its domain and all constants c_1 and c_2 we have

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2).$$

It's often convenient to separate this into two conditions

$$(L) \qquad \mathcal{L}(u_1 + u_2) = \mathcal{L}(u_1) + \mathcal{L}(u_2) \ \text{ and } \ \mathcal{L}(cu) = c\mathcal{L}(u)$$

which both must be satisfied for linear $\mathcal{L}$.

A differential operator $\mathcal{D}(u)$ is some function of u and derivatives of u. Two examples of differential operators are

$$(i) \ \ \mathcal{D}(u) \equiv e^x\frac{du}{dx} + \frac{d^3u}{dx^3}, \quad (ii) \ \ \mathcal{D}(u) \equiv u\frac{du}{dx} + \sin(x)\frac{d^2u}{dx^2}.$$

The *order* of $\mathcal{D}$ is the order of the highest derivative in $\mathcal{D}$. (i) above is a third order differential operator, whereas (ii) is second order.

Recall from calculus that the derivative operator is linear, i.e.

$$\frac{d}{dx}(u_1 + u_2) = \frac{du_1}{dx} + \frac{du_2}{dx} \ \text{ and } \ \frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Clearly, so is the derivative of any order n

$$\frac{d^n}{dx^n}(u_1 + u_2) = \frac{d^nu_1}{dx^n} + \frac{d^nu_2}{dx^n} \ \text{ and } \ \frac{d^n}{dx^n}(cu) = c\frac{d^nu}{dx^n}.$$

Use these to see that $\mathcal{D}$ defined in (i) above is linear:

$$\mathcal{D}(u_1 + u_2) = e^x\frac{d(u_1 + u_2)}{dx} + \frac{d^3(u_1 + u_2)}{dx^3} = e^x\left(\frac{du_1}{dx} + \frac{du_2}{dx}\right) + \left(\frac{d^3u_1}{dx^3} + \frac{d^3u_2}{dx^3}\right)$$

$$= e^x\frac{du_1}{dx} + \frac{d^3u_1}{dx^3} + e^x\frac{du_2}{dx} + \frac{d^3u_2}{dx^3} = \mathcal{D}(u_1) + \mathcal{D}(u_2)$$

and

$$\mathcal{D}(cu) = e^x\frac{d(cu)}{dx} + \frac{d^3(cu)}{dx^3} = e^xc\frac{du}{dx} + c\frac{d^3u}{dx^3} = c\left(e^x\frac{du}{dx} + \frac{d^3u}{dx^3}\right) = c\mathcal{D}(u).$$

On the other hand, $\mathcal{D}$ defined in (ii) is not linear. This is true if <u>either</u> condition in (L) is invalid. For example, since in general

$$\mathcal{D}(cu) = (cu)\frac{d(cu)}{dx} + \sin(x)\frac{d^2(cu)}{dx^2} = c^2u\frac{du}{dx} + c\sin(x)\frac{d^2u}{dx^2} \neq c\mathcal{D}(u),$$

you can conclude that this differential operator is not linear.

A *differential equation* can be written as $\mathcal{D}(u) = 0$. When this can be written in the particular form

$$\mathcal{L}(u) = f(x)$$

where $f(x)$ is a given function of x and $\mathcal{L}(u)$ is a linear differential operator, we call this a *linear differential equation.* When $f(x) \equiv 0$ we say the differential equation is *linear and homogeneous.*

For example, consider the differential equation

$$\frac{d^3u}{dx^3} + e^x \frac{du}{dx} + x = 0 \quad \Longrightarrow \quad \frac{d^3u}{dx^3} + e^x \frac{du}{dx} = -x,$$

which can be written as

$$\mathcal{L}(u) = -x \quad \text{where} \quad \mathcal{L}(u) = \frac{d^3u}{dx^3} + e^x \frac{du}{dx}.$$

Since $\mathcal{L}$ is linear as shown above, the given differential equation is linear but in this case not homogeneous.

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1. Determine which of the following differential operators $\mathcal{D}(y)$ (the dependent variable here is y) are linear and which are not. Also, state $\mathcal{D}$'s order.

(a) $\mathcal{D}(y) \equiv y\dfrac{dy}{dx}$ (c) $\mathcal{D}(y) \equiv x^2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}$

(b) $\mathcal{D}(y) \equiv y + \dfrac{dy}{dx}$ (d) $\mathcal{D}(y) \equiv \dfrac{dy}{dx} + y^2$

2. Which are linear and homogeneous differential equations? Which are linear but not homogeneous? Which are nonlinear differential equations? State the order of each.

(a) $\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} + y = 0$ (c) $e^x\dfrac{d^3y}{dx^3} = y$

(b) $xy + \dfrac{dy}{dx} + \sin x = 0$ (d) $\dfrac{d^2y}{dx^2} = x^2y + \dfrac{dy}{dx}$

Consider the following first order differential equation

$$2x^2\frac{du}{dx} - u^2 = x^2.$$

Somehow I was able to figure out that

$$u(x) = x\left(1 - \frac{2}{\log|x|}\right)$$

is a solution. This fact can be verified by plugging u and its derivative back into the differential equation to find

$$\begin{aligned} u^2 + x^2 &= 2x^2\left[\frac{2}{(\log|x|)^2} - \frac{2}{\log|x|} + 1\right] \\ 2x^2\frac{du}{dx} &= 2x^2\left[1 - \frac{2}{\log|x|} + \frac{2}{(\log|x|)^2}\right] \end{aligned} \quad \Rightarrow \quad 2x^2\frac{du}{dx} = u^2 + x^2.$$

In general, a first order differential equation will have a one-parameter family of solutions. A second order differential equation will have a two-parameter family of solutions, etc. The so-called *general solution* to the previous example is in fact given by

$$u(x) = x\left(1 - \frac{2}{\log|x| + c}\right),$$

where c can be any constant. See exercise 7 below.

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3. Verify that the given function is a solution to the corresponding differential equation. (Just plug it in and see what you get.)

   (a) $u(x) = e^{-x}$ solves $\dfrac{du}{dx} + u = 0$

   (b) $u(x) = -\dfrac{x}{\log x}$ solves $x^2\dfrac{du}{dx} = u^2 + xu$

   (c) $u(x) = 2\arctan(e^x)$ solves $\dfrac{du}{dx} = \sin u$

4. Verify that the given function is a solution to the corresponding differential equation.

   (a) $u(x) = xe^x$ solves $\dfrac{d^2u}{dx^2} - 2\dfrac{du}{dx} + u = 0$

   (b) $u(x) = e^{e^x}$ solves $u\dfrac{d^2u}{dx^2} - \left(\dfrac{du}{dx}\right)^2 - u^2\log u = 0$

   (c) $u(x) = e^{x/2}$ solves $u\dfrac{d^2u}{dx^2} + \left(\dfrac{du}{dx}\right)^2 - \dfrac{1}{2}u^2 = 0$

If a differential equation is linear and homogeneous, that is $\mathcal{L}(u) = 0$ where $\mathcal{L}$ is a linear differential operator, then any linear combination of solutions is also a solution. That is, given that

$$\mathcal{L}(u_1) = 0 \text{ and } \mathcal{L}(u_2) = 0 \quad \Rightarrow \quad u_g = c_1u_1 + c_2u_2 \text{ also solves } \mathcal{L}(u_g) = 0.$$

Clearly this is true by linearity

$$\mathcal{L}(u_g) = \mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2) = c_1\,0 + c_2\,0 = 0.$$

This is often called the *principle of superposition.* Please note however, this form of superposition only applies to linear and homogeneous problems.

Superposition can often be exploited to solve certain ordinary differential equations (abbreviated ODEs) with various possible side conditions. For example

$$u_1(x) = \sin x \text{ and } u_2(x) = \cos x \text{ both solve } \frac{d^2u}{dx^2} + u = 0.$$

Let's use these to solve what's called an initial value problem

$$\frac{d^2u}{dx^2} + u = 0 \quad \text{where } u \text{ must also satisfy} \quad u(0) = 1,\ u_x(0) = 2.$$

Since the ODE is linear and homogeneous,

$$u(x) = c_1 \sin x + c_2 \cos x$$

solves the ODE for arbitrary constants c_1 and c_2. Now let's see if we can determine explicit values for c_1 and c_2 in order to satisfy the given initial conditions

$$1 = u(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \quad \text{and} \quad 2 = u_x(0) = c_1 \cos(0) - c_2 \sin(0) = c_1$$

$$\Rightarrow \quad c_2 = 1 \quad \text{and} \quad c_1 = 2 \quad \Rightarrow \quad u(x) = 2\sin x + \cos x.$$

And so there is the sought for solution to our initial value problem.

Next let's consider a linear but this time an inhomogeneous problem

$$\mathcal{L}(u) = f(x).$$

Suppose again we can find u_1 and u_2 which solve the homogeneous linear differential equation $\mathcal{L}(u_1) = \mathcal{L}(u_2) = 0$. Suppose we can also find what we will later call a *particular solution* which solves $\mathcal{L}(u_p) = f(x)$. Then again by linearity

$$u_g(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x) \text{ also solves } \mathcal{L}(u_g) = f(x).$$

Note how we've added in the particular solution into u_g. As an example, check that

$$u_p(x) = x \quad \text{is a particular solution to} \quad \frac{d^2u}{dx^2} + u = x.$$

Therefore,

$$u(x) = c_1 \sin x + c_2 \cos x + x \quad \text{also solves} \quad \frac{d^2u}{dx^2} + u = x$$

for arbitrary constants c_1 and c_2. This can be used to solve e.g. the initial value problem with conditions $u(0) = 1$, $u_x(0) = 2$, and we obtain the solution

$$u(x) = \sin x + \cos x + x.$$

For a nonlinear ODE, there really isn't an easy way to piece together the general solution as done above for the linear problem. Consider

$$\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 = 0 \quad \text{for which somehow I computed} \quad u(x) = -\log|x - c_1| + c_2.$$

Here's your two-parameter family of solutions. This can be used to solve the example initial value problem with $u(0) = 1$, $u_x(0) = 2$:

$$1 = u(0) = -\log|-c_1| + c_2, \quad 2 = u_x(0) = 1/c_1 \quad \Rightarrow \quad c_1 = 1/2,\ c_2 = 1 + \log 1/2,$$

and we obtain the initial value problem's solution

$$u(x) = -\log|x - 1/2| + 1 + \log 1/2.$$

You might want to note for later consideration that this solution blows up (its value runs off to infinity) as x approaches $1/2$ from below. The solution can not be continued beyond this point.

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5. The second order ODE

$$\frac{d^2u}{dx^2} - u = 0 \quad \text{has two solutions:} \quad u_1(x) = \sinh x \quad \text{and} \quad u_2(x) = \cosh x.$$

Note the equation is linear and homogeneous. Use u_1 and u_2 to solve the ODE with the following initial conditions.

(a) $u(0) = 1,\ u_x(0) = 2$ (b) $u(0) = 2,\ u_x(0) = 1$

Now use u_1 and u_2 to solve the ODE with the following boundary conditions.

(c) $u(0) = 0,\ u(1) = 1$ (d) $u(0) = 1,\ u(1) = 0$ (e) $u_x(0) = 1,\ u_x(1) = 2$

6. The second order linear but inhomogeneous ODE

$$\frac{d^2u}{dx^2} - u = x \quad \text{has a particular solution:} \quad u_p(x) = -x.$$

Use u_1 and u_2 from the previous exercise together with this particular solution to solve the ODE with the following initial conditions.

(a) $u(0) = 1,\ u_x(0) = 2$ (b) $u(0) = 2,\ u_x(0) = 1$

Solve the inhomogeneous ODE with the following boundary conditions.

(c) $u(0) = 0,\ u(1) = 1$ (d) $u(0) = 1,\ u(1) = 0$ (e) $u_x(0) = 1,\ u_x(1) = 2$

7. Recall I claimed the general solution to the first order ODE

$$2x^2\frac{du}{dx} - u^2 = x^2 \quad \text{is} \quad u(x) = x\left(1 - \frac{2}{\log|x| + c}\right).$$

(a) Plug the solution into the ODE to verify it's valid.

(b) Solve the initial value problem with $u(1) = 0$.

(c) Solve the initial value problem with $u(1) = 1$.

I got: (b) $u(x) = x(1 - 2/(\log x + 2))$ and (c) $u(x) = x$.

8. Recall the general solution to

$$\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 = 0 \quad \text{is} \quad u(x) = -\log|x - c_1| + c_2.$$

(a) Plug the solution into the ODE to verify it's valid.

(b) Solve the initial value problem with $u(0) = 0$, $u_x(0) = 1$.

(c) Solve the initial value problem with $u(0) = 1$, $u_x(0) = 0$.

I got: (b) $u(x) = -\log(1 - x)$ and (c) $u(x) = 1$.
