2×2 Non-diagonalizable Systems

I'd like to briefly discuss how to evaluate analytic functions of non-diagonalizable matrices in closed form. The general $n \times n$ case can be quite involved, so I will restrict my attention to the 2×2 case only.

Consider the following matrix which has only one eigenvalue and whose corresponding eigenspace is one dimensional.

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad \lambda = 2, \quad \mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This matrix therefore can not be diagonalized. But it is similar to a matrix that is "almost" diagonal

$$S^{-1}AS = J_\lambda \equiv \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

J_λ is this A's *Jordan canonical form*; see [†].

Here's a recipe to determine S for a 2×2 non-diagonalizable matrix A. Let $\mathbf{g} \neq \mathbf{0}$ be <u>any</u> vector that is **not** an eigenvector of A. From this, it can be shown that $\mathbf{r} \equiv (A - \lambda I)\,\mathbf{g}$ is an eigenvector. For A given above:

$$\text{Take for example } \mathbf{g} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{r} = \left(\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Next, build S using $(A - \lambda I)\,\mathbf{g}$ as its first column and $\mathbf{g}$ as its second. For the given A

$$S = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \quad \Rightarrow \quad S^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

This S will yield $S^{-1}AS = J_\lambda$. For the example matrix

$$S^{-1}AS = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J_\lambda.$$

Please note, this recipe works for 2×2 non-diagonalizable matrices. It does <u>not</u> apply to diagonalizable matrices!

Now recall for an analytic function f

$$f(x) = \sum_{n=0}^{\infty} \gamma_n x^n \quad \Rightarrow \quad f(At) \equiv \sum_{n=0}^{\infty} \gamma_n A^n t^n.$$

Suppose A is 2×2 but not diagonalizable. Just as done earlier

$$A^n = (SJ_\lambda S^{-1})^n = S(J_\lambda)^n S^{-1}. \quad \Rightarrow \quad f(At) = S \left(\sum_{n=0}^{\infty} \gamma_n (J_\lambda)^n t^n \right) S^{-1}$$

[†] `https://en.wikipedia.org/wiki/Jordan_normal_form`

Here's where things are a bit different. The binomial theorem will show, for $n \geq 1$, that

$$(J_\lambda)^n = (\lambda I + N)^n = \lambda^n I + n\lambda^{n-1} N \quad \text{where } N \text{ is given by } \; N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and so

$$f(At) = S \left(\sum_{n=0}^{\infty} \gamma_n \lambda^n t^n \, I + \sum_{n=1}^{\infty} \gamma_n n \lambda^{n-1} t^n \, N \right) S^{-1}.$$

Moreover, one should easily see that

$$\sum_{n=0}^{\infty} \gamma_n \lambda^n t^n = f(\lambda t) \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n n \lambda^{n-1} t^n = t f'(\lambda t).$$

Therefore

$$f(At) = f(\lambda t) \, I + t f'(\lambda t) \, S N S^{-1}.$$

For example,

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \Rightarrow \quad e^{J_\lambda t} = e^{\lambda t} I + t e^{\lambda t} N = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

---

1. Consider the non-diagonalizable matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ from above.
   (a) Find another matrix S (different from mine) which casts A into its canonical form.
   (b) Determine e^{At}.
   (c) Confirm that $de^{At}/dt = Ae^{At}$.

Answer for (b) $e^{At} = e^{2t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$

2. Consider the matrix $A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$.
   (a) Show why A is not diagonalizable.
   (b) What is A's Jordan form J_λ?
   (c) Determine S so the $S^{-1} A S = J_\lambda$.

3. Use your results from exercise 1 to solve the following initial value problem.

$$\begin{cases} \dfrac{dx}{dt} = 3x + y, & x(0) = 1, \\[2mm] \dfrac{dy}{dt} = -x + y, & y(0) = 2. \end{cases}$$

4. Use your results from exercise 2 to solve the following initial value problem.

$$\begin{cases} \dfrac{dx}{dt} = -x + 4y, \quad x(0) = 1, \\ \dfrac{dy}{dt} = -x + 3y, \quad y(0) = 2. \end{cases}$$

5. Compute $\sin(At)$ where the matrix A is given in exercise 1.

Answer: $\sin(At) = \begin{pmatrix} \sin(2t) + t\cos(2t) & t\cos(2t) \\ -t\cos(2t) & \sin(2t) - t\cos(2t) \end{pmatrix}.$
