2 × 2 Non-diagonalizable Systems

I’d like to briefly discuss how to evaluate analytic functions of non-diagonalizable matrices in closed form. The general $n \times n$ case can be quite involved, so I will restrict my attention to the $2 \times 2$ case only.

Consider the following matrix which has only one eigenvalue and whose corresponding eigenspace is one dimensional.

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad \lambda = 2, \quad r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$  

This matrix therefore can not be diagonalized. But it is similar to a matrix that is "almost" diagonal

$$S^{-1}AS = J_\lambda \equiv \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$  

$J_\lambda$ is this $A$’s Jordan canonical form; see [†].

Here’s a recipe to determine $S$ for a $2 \times 2$ non-diagonalizable matrix $A$. Let $g \neq 0$ be any vector that is not an eigenvector of $A$. From this, it can be shown that $r \equiv (A - \lambda I)g$ is an eigenvector. For $A$ given above:

Take for example $g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ \Rightarrow $r = \begin{pmatrix} (3 & 1) - 2(1 & 0) \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

Next, build $S$ using $(A - \lambda I)g$ as its first column and $g$ as its second. For the given $A$

$$S = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix} \Rightarrow S^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}.$$  

This $S$ will yield $S^{-1}AS = J_\lambda$. For the example matrix

$$S^{-1}AS = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J_\lambda.$$  

Please note, this recipe works for $2 \times 2$ non-diagonalizable matrices. It does not apply to diagonalizable matrices!

Now recall for an analytic function $f$

$$f(x) = \sum_{n=0}^{\infty} \gamma_n x^n \quad \Rightarrow \quad f(At) = \sum_{n=0}^{\infty} \gamma_n A^n t^n.$$  

Suppose $A$ is $2 \times 2$ but not diagonalizable. Just as done earlier

$$A^n = (SJ_\lambda S^{-1})^n = S(J_\lambda)^n S^{-1} \quad \Rightarrow \quad f(At) = S \left( \sum_{n=0}^{\infty} \gamma_n (J_\lambda)^n t^n \right) S^{-1}.$$  

Here’s where things are a bit different. The binomial theorem will show, for \( n \geq 1 \), that
\[
(J_\lambda)^n = (\lambda I + N)^n = \lambda^n I + n\lambda^{n-1}N \quad \text{where} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
and so
\[
f(At) = S \left( \sum_{n=0}^{\infty} \gamma_n \lambda^n t^n I + \sum_{n=1}^{\infty} \gamma_n n\lambda^{n-1}t^n N \right) S^{-1}.
\]
Moreover, one should easily see that
\[
\sum_{n=0}^{\infty} \gamma_n \lambda^n t^n = f(\lambda t) \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n n\lambda^{n-1}t^n = t f'(\lambda t).
\]
Therefore
\[
f(At) = f(\lambda t) I + tf'(\lambda t) SNS^{-1}.
\]
For example,
\[
J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \Rightarrow \quad e^{J_\lambda t} = e^{\lambda t} I + te^{\lambda t} N = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.
\]

1. Consider the non-diagonalizable matrix \( A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \) from above.
   (a) Find another matrix \( S \) (different from mine) which casts \( A \) into its canonical form.
   (b) Determine \( e^{At} \).
   (c) Confirm that \( de^{At}/dt = Ae^{At} \).

   Answer for (b) \( e^{At} = e^{2t} \begin{pmatrix} 1 + t & t \\ -t & 1 - t \end{pmatrix} \)

2. Consider the matrix \( A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \).
   (a) Show why \( A \) is not diagonalizable.
   (b) What is \( A \)'s Jordan form \( J_\lambda \)?
   (c) Determine \( S \) so the \( S^{-1}AS = J_\lambda \).

3. Use your results from exercise 1 to solve the following initial value problem.
\[
\begin{cases}
\frac{dx}{dt} = 3x + y, \quad x(0) = 1, \\
\frac{dy}{dt} = -x + y, \quad y(0) = 2.
\end{cases}
\]
4. Use your results from exercise 2 to solve the following initial value problem.

\[
\begin{aligned}
\frac{dx}{dt} &= -x + 4y, \quad x(0) = 1, \\
\frac{dy}{dt} &= -x + 3y, \quad y(0) = 2.
\end{aligned}
\]

5. Compute \(\sin(At)\) where the matrix \(A\) is given in exercise 1.

Answer: \(\sin(At) = \begin{pmatrix}
\sin(2t) + t\cos(2t) & t\cos(2t) \\
-t\cos(2t) & \sin(2t) - t\cos(2t)
\end{pmatrix}\).