Second Order Linear Equations (Part I)

A second order linear ODE, written here in standard form, looks like

$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = f(x),$$

where a(x), b(x) and f(x) are given functions of x. I'll call this equation homogeneous when $f(x) \equiv 0$. I'll call this equation constant coefficient when a(x) and b(x) are constants.

Suppose, somehow, I could *factor* the standard form differential operator as follows,

(F1)
$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = \left(\frac{d}{dx} - p(x)I\right)\left(\frac{d}{dx} - q(x)I\right)u.$$

(The notation I above denotes the identity operator.) Carry out the indicated operations on the right to get

$$\left(\frac{d}{dx} - p(x)I\right) \left(\frac{d}{dx} - q(x)I\right) u$$

= $\frac{d}{dx} \left(\frac{du}{dx} - q(x)u\right) - p(x) \left(\frac{du}{dx} - q(x)u\right)$
= $\frac{d^2u}{dx^2} - (p(x) + q(x))\frac{du}{dx} + (p(x)q(x) - q'(x))u$

Therefore, to obtain the factorization (F1), we must determine p(x) and q(x) so that

(F2)
$$-(p(x) + q(x)) \equiv a(x)$$
 and $(p(x)q(x) - q'(x)) \equiv b(x)$.

Solving for p and q is an almost impossible task to do generally except in certain special (but very important) cases. I'll discuss these later.

Given that a second order linear equation is already factored, solving it simply reduces to solving two first order linear ODEs. Here's an example. Consider

$$\frac{d^2u}{dx^2} - u = \left(\frac{d}{dx} - I\right)\left(\frac{d}{dx} + I\right)u = x.$$

Call v = (d/dx + I) u and solve

$$\frac{dv}{dx} - v = x \quad \Rightarrow \quad e^x \frac{d}{dx} \left(e^{-x} v \right) = x \quad \Rightarrow \quad v = c_1 e^x - (x+1).$$

Finally, use v just computed to solve for u

$$\frac{du}{dx} + u = v \quad \Rightarrow \quad e^{-x} \frac{d}{dx} (e^x u) = c_1 e^x - (x+1) \quad \Rightarrow \quad u = c_2 e^{-x} + \frac{1}{2} c_1 e^x - x.$$

Redefine constant c_1 to see

$$u = c_1 e^x + c_2 e^{-x} - x$$
 is the general solution to $\frac{d^2 u}{dx^2} - u = x$.

Notice that the second order equation's general solution involves two free parameters, c_1 and c_2 .

Here's a more interesting variable coefficient example. Let's start with the factored equation, and from it work backwards to determine the standard form equation.

$$\left(\frac{d}{dx} - \frac{1}{x}I\right)\left(\frac{d}{dx} + \frac{1}{x}I\right)u = \frac{d^2u}{dx^2} + \frac{d}{dx}\left(\frac{1}{x}u\right) - \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = \frac{d^2u}{dx^2} - \frac{2}{x^2}u.$$

So, to solve the second order variable coefficient problem

$$\frac{d^2u}{dx^2} - \frac{2}{x^2}u = 0,$$

first let v = (d/dx + 1/x I) u and solve for v

$$\frac{dv}{dx} - \frac{1}{x}v = 0 \quad \Rightarrow \quad x\frac{d}{dx}\left(\frac{1}{x}v\right) = 0 \quad \Rightarrow \quad v = c_1x,$$

then solve for u

$$\frac{du}{dx} + \frac{1}{x}u = v \quad \Rightarrow \quad \frac{1}{x}\frac{d}{dx}(xu) = c_1x \quad \Rightarrow \quad u = \frac{1}{3}c_1x^2 + c_2x^{-1}.$$

Of course I've cheated on this example by starting with the factored equation.

1. Verify that I've correctly factored the given standard form differential operator, then determine the general solution to the linear and homogeneous ODE.

(a)
$$\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+2I\right)u = \frac{d^2u}{dx^2}+3\frac{du}{dx}+2u = 0$$

(b) $\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+I\right)u = \frac{d^2u}{dx^2}+2\frac{du}{dx}+u = 0$
(c) $\left(\frac{d}{dx}\right)\left(\frac{d}{dx}-\frac{1}{x}I\right)u = \frac{d^2u}{dx^2}-\frac{1}{x}\frac{du}{dx}+\frac{1}{x^2}u = 0$
(d) $\left(\frac{d}{dx}+2xI\right)\left(\frac{d}{dx}\right)u = \frac{d^2u}{dx^2}+2x\frac{du}{dx}=0$
et (d) use $\int e^{-x^2} dx = \sqrt{\pi} \operatorname{erf}(x)$

On part (d) use $\int e^{-x} dx = \frac{\sqrt{n}}{2} \operatorname{erf}(x)$. See e.g. https://en.wikipedia.org/wiki/Error_function.

2. Use the factorizations given in the previous exercise to find the general solutions to the following linear but this time inhomogeneous ODEs.

(a)
$$\frac{d^2u}{dx^2} + 3\frac{du}{dx} + 2u = x$$
 (c) $\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} + \frac{1}{x^2}u = 1$
(b) $\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = e^{-x}$ (d) $\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = x$

Now, let me show you how to perform factorization (F1) in two special cases. In both cases I'll use requirement (F2), but here I'll decouple the two equations for p(x) and q(x) and write them as follows.

(F2*)
$$q'(x) = -(q^2(x) + a(x)q(x) + b(x))$$
 and $p(x) = -(q(x) + a(x))$.

Note, once q(x) is determined from the left equation, p(x) follows from the right.

The constant coefficient equation:

First, suppose the coefficients a(x) and b(x) are constant, i.e.

$$a(x) \equiv a, \quad b(x) \equiv b.$$

In this case we can solve $(F2^*)$ by assuming q is constant.

$$\begin{split} q &\equiv r = const \\ q' &= -(q^2 + aq + b) \quad \Rightarrow \quad 0 = -(r^2 + ar + b) \quad \Rightarrow \quad q = r = \frac{1}{2} \left(-a \pm \sqrt{a^2 - 4b} \right), \\ p &= -(q + a) \qquad \Rightarrow \quad p = -(r + a) \qquad \Rightarrow \quad p = \frac{1}{2} \left(-a \mp \sqrt{a^2 - 4b} \right). \end{split}$$

The constant coefficient factorization is therefore

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = \left(\frac{d}{dx} - r_1I\right)\left(\frac{d}{dx} - r_2I\right)u,$$

where r_1 and r_2 are the two roots of this ODE's characteristic polynomial,

(CC)
$$r^2 + ar + b = 0$$

Many applications involve second order constant coefficient ODEs. You'll see them over and over in this course, and you'll also see them in many upper level engineering courses. I advise you give this constant coefficient factorization formula careful thought.

The Cauchy-Euler differential equation:

A variable coefficient equation whose standard form coefficients take the form

$$a(x) \equiv \frac{a}{x}, \ b(x) \equiv \frac{b}{x^2}$$
 (a and b are constants)

is called a Cauchy-Euler equation. This equation arises when solving the Laplace partial differential equation in two dimensional circular symmetry by separation of variables; (you'll use it in Math 3363). In this case (F2^{*}) can be solved by taking q(x) = r/x and determining the constant r.

$$\begin{aligned} q(x) &= r/x, \ q' = -(q^2 + (a/x)q + (b/x^2)) \implies -r = -(r^2 + ar + b) \\ &\Rightarrow \ r = \frac{1}{2} \left(-(a-1) \pm \sqrt{(a-1)^2 - 4b} \right). \end{aligned}$$

From this, solve for p(x)

$$p(x) = -(q(x) + a/x) \quad \Rightarrow \quad p(x) = \frac{-\frac{1}{2}\left(-(a-1) \pm \sqrt{(a-1)^2 - 4b} + 2a\right)}{x}$$
$$= \frac{\frac{1}{2}\left(-(a-1) \mp \sqrt{(a-1)^2 - 4b}\right) - 1}{x}.$$

The Cauchy-Euler factorization can also be written in a fairly compact form.

$$\frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx} + \frac{b}{x^2}u = \left(\frac{d}{dx} - \frac{(r_1 - 1)}{x}I\right)\left(\frac{d}{dx} - \frac{r_2}{x}I\right)u,$$

where r_1 and r_2 are the two roots of the C-E characteristic polynomial,

(CE)
$$r(r-1) + ar + b = 0.$$

3. Factor the following differential operators. Check your work.

| (a) | $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u$ | (d) | $\frac{d^2u}{dx^2}$ – | $\frac{1}{x}\frac{du}{dx} + \frac{1}{x^2}u$ |
|-----|---|-----|-----------------------|---|
| (b) | $\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u$ | (e) | ${d^2 u\over dx^2}$ – | $\frac{2}{x}\frac{du}{dx} + \frac{2}{x^2}u$ |
| (c) | $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + 2u$ | (f) | ${d^2 u\over dx^2}$ – | $\frac{1}{x}\frac{du}{dx} + \frac{2}{x^2}u$ |

Factorization of both the constant coefficient and Cauchy-Euler equations require determining the roots of a quadratic characteristic polynomial; see (CC) and (CE) above. Three cases must be considered.

(Case 1) There are two real and distinct roots.

(Case 2) There is only one root.

(Case 3) The two roots are complex.

We're going to focus on the homogeneous problem here, the inhomogeneous problem will be dealt with on the next homework assignment.

The constant coefficient equation:

We'll find the general solution to

$$\frac{d^2u}{dx^2} + a \frac{du}{dx} + b u = 0, \quad (a, b \text{ are constant})$$

whose characteristic polynomial, see (CC) above, has roots

$$r^2 + ar + b = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left(-a \pm \sqrt{a^2 - 4b} \right).$$

Factor and solve as done earlier to conclude

$$\left(\frac{d}{dx} - r_1 I\right) \left(\frac{d}{dx} - r_2 I\right) u = 0 \quad \Rightarrow \quad u = \frac{c_1}{r_1 - r_2} e^{r_1 x} + c_2 e^{r_2 x}$$

In case 1 and case 3 $r_1 \neq r_2$, so the given solution is valid. Furthermore, the constant c_1 can be redefined $c_1/(r_1 - r_2) \rightarrow c_1$. The solution above is however <u>not</u> valid in case 2, $r_1 = r_2$, since division by zero is never allowed. It will be treated last.

The roots are real and distinct when $a^2 - 4b > 0$. Write $r_1 = \alpha + \beta$ and $r_2 = \alpha - \beta$ by taking $\alpha \equiv -a/2$ and $\beta \equiv \sqrt{a^2 - 4b}/2$. So we can write our solution as

$$u = e^{\alpha x} \left(c_1 e^{\beta x} + c_2 e^{-\beta x} \right).$$

Redefine constants again, $c_1 \to (\tilde{c}_1 + \tilde{c}_2)/2$ and $c_2 \to (\tilde{c}_1 - \tilde{c}_2)/2$, to see this can be further written as

(case 1)
$$u = e^{\alpha x} \left(\tilde{c}_1 \cosh(\beta x) + \tilde{c}_2 \sinh(\beta x) \right).$$

The roots are complex when $a^2 - 4b < 0$. In this case, write $\beta \equiv \sqrt{4b - a^2}/2$ (what's inside the square root is positive) so $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. The solution becomes

$$u = e^{\alpha x} \left(c_1 e^{i\beta x} + c_2 e^{-i\beta x} \right).$$

Redefine constants again, this time $c_1 \rightarrow (\tilde{c}_1 - i\tilde{c}_2)/2$ and $c_2 \rightarrow (\tilde{c}_1 + i\tilde{c}_2)/2$, together with Euler's formula

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

to get

(case 3)
$$u = e^{\alpha x} \left(\tilde{c}_1 \cos(\beta x) + \tilde{c}_2 \sin(\beta x) \right).$$

There is only one root when $a^2 - 4b = 0$ in which case $r_1 = r_2 = -a/2 \equiv \alpha$. In this special case,

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + (a/2)^2 u = \left(\frac{d}{dx} - \alpha I\right)\left(\frac{d}{dx} - \alpha I\right)u = 0,$$

and as before set $v = du/dx - \alpha u$, solve for v, then solve for u

$$\frac{dv}{dx} - \alpha v = 0 \quad \Rightarrow \quad v = c_1 e^{\alpha x} \quad \Rightarrow \quad \frac{du}{dx} - \alpha u = v = c_1 e^{\alpha x} \quad \Rightarrow \quad u = (c_1 x + c_2) e^{\alpha x}.$$

So when the characteristic polynomial has only one root, $r = -a/2 \equiv \alpha$, the ODE's general solution is

$$(case 2) u = (c_1 x + c_2) e^{\alpha x}$$

Here are two examples.

The constant coefficient homogeneous equation has characteristic polynomial

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + 3u = 0, \quad r^2 + 2r + 3 = 0.$$

The roots are found by the quadratic formula

$$r_{\pm} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = -1 \pm \sqrt{1 - 3} = -1 \pm \sqrt{2} i.$$

(This is case 3.) Therefore, its general solution is

$$u = e^{-x} \left(c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) \right).$$

The constant coefficient homogeneous equation has characteristic polynomial

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 0, \quad r^2 + 2r + 1 = 0.$$

Here, the quadratic formula gives

$$r_{\pm} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1}}{2} = -1 \pm 0.$$

(This is case 2.) Therefore, its general solution is

$$u = e^{-x} \left(c_1 x + c_2 \right).$$

The Cauchy-Euler differential equation:

We'll now find the general solution to

$$\frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx} + \frac{b}{x^2}u = 0, \quad (a, b \text{ are constant})$$

whose characteristic polynomial, see (CE) above, has roots

$$r(r-1) + ar + b = 0 \implies r_{\pm} = \frac{1}{2} \left(-(a-1) \pm \sqrt{(a-1)^2 - 4b} \right)$$

Take r_1 as either root and r_2 the other, and then factor

$$\left(\frac{d}{dx} - \frac{(r_1 - 1)}{x}I\right)\left(\frac{d}{dx} - \frac{r_2}{x}I\right)u = 0$$

Solve as usual, set $v = du/dx - (r_2/x)u$, solve for v, then solve for u

$$\frac{dv}{dx} - \frac{(r_1 - 1)}{x}v = 0 \qquad \Rightarrow \quad v = c_1 x^{r_1 - 1}$$
$$\frac{du}{dx} - \frac{r_2}{x}u = v = c_1 x^{r_1 - 1} \qquad \Rightarrow \quad u = \frac{c_1}{r_1 - r_2} x^{r_1} + c_2 x^{r_2}$$

As we saw earlier, this is <u>not</u> valid when $r_1 = r_2$. So, given $r_2 = r_1$, what changes is

$$\frac{du}{dx} - \frac{r_1}{x}u = v = c_1 x^{r_1 - 1} \quad \Rightarrow \quad x^{r_1} \frac{d}{dx} \left(x^{-r_1} u \right) = c_1 x^{r_1 - 1} \quad \Rightarrow \quad u = \left(c_1 \log(x) + c_2 \right) x^{r_1} + c_2 x^{r_1$$

To summarize, when the Cauchy-Euler characteristic polynomial has two distinct real roots, say r_1 and r_2 , the general solution to the homogeneous problem is

$$u = c_1 x^{r_1} + c_2 x^{r_2}$$

When there is only one root, say $r_2 = r_1$, the general solution is

$$u = (c_1 \log(x) + c_2) x^{r_1}$$

I know of no *real* applications of Cauchy-Euler whose characteristic polynomial has complex roots. Therefore, I won't treat this case here. (It's not hard to do however. Give it a try if you've got some spare time.)

Here's an example of Cauchy-Euler you'll see, Math 3363 or other PDE courses, when doing separation of variables applied to the Laplacian in circular symmetry. Solve

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{n^2}{x^2}u = 0$$

where $n = 0, 1, 2, \ldots$ Its characteristic polynomial and roots are

$$r(r-1) + r - n^2 = 0 \quad \Rightarrow \quad r_{\pm} = 0 \pm n.$$

There is only one distinct root when n = 0, so in this case

$$u = c_1 \log(x) + c_2.$$

There are two distinct real roots when $n \neq 0$, so in this case

$$u = c_1 x^{-n} + c_2 x^n.$$

4. Determine the general solution to the following constant coefficient and homogeneous equations.

(a)
$$\frac{d^2u}{dx^2} - u = 0$$
 (d) $\frac{d^2u}{dx^2} + u = 0$
(b) $\frac{d^2u}{dx^2} - \frac{du}{dx} - 2u = 0$ (e) $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + 2u = 0$
(c) $\frac{d^2u}{dx^2} = 0$ (f) $\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 0$

5. Solve the following IVPs.

(a)
$$\frac{d^2u}{dx^2} = 0$$
 with $u(0) = 1$, $u'(0) = 2$
(b) $\frac{d^2u}{dx^2} - \frac{du}{dx} + 2u = 0$ with $u(0) = 1$, $u'(0) = 2$

6. Solve the following boundary value problems, (BVPs).

Dirichlet (a)
$$\frac{d^2u}{dx^2} - u = 0$$
 with $u(0) = 1$, $u(1) = 2$
Neumann (b) $\frac{d^2u}{dx^2} - u = 0$ with $u'(0) = 1$, $u'(1) = 2$

7. Here are a couple of Cauchy-Euler problems. Find the general solution.

(a)
$$\frac{d^2u}{dx^2} - \frac{3}{x}\frac{du}{dx} + \frac{3}{x^2}u = 0$$
 (b) $\frac{d^2u}{dx^2} - \frac{3}{x}\frac{du}{dx} + \frac{4}{x^2}u = 0$

8. When solving Laplace's PDE on an annular region, the following Dirichlet–type BVPs must be solved. Please solve.

(a)
$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0$$
 with $u(1) = a_0, \ u(2) = b_0$
(b) $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{n^2}{x^2} = 0$ with $u(1) = a_n, \ u(2) = b_n \ (n = 1, 2, ...)$

FYI: a_n , b_n are constants, specifically Fourier coefficients.