How to Easily Solve Certain Second Order Linear ODEs

The Method of Guessing and Duhamel's Principle

Here we consider finding the general solution to

$$\mathcal{L}(u) = f(x)$$

where $\mathcal{L}(u)$ is a linear second order differential operator. The solution u can be decomposed into two parts

$$u = u_h + u_p \quad \text{where} \quad \mathcal{L}(u_h) = 0, \quad \text{and} \quad \mathcal{L}(u_p) = f(x).$$

u_h is called the *homogeneous solution* and u_p is called a *particular solution*. Clearly $u_h + u_p$ solves the inhomogeneous ODE since

$$\mathcal{L}(u) = \mathcal{L}(u_h + u_p) = \mathcal{L}(u_h) + \mathcal{L}(u_p) = 0 + f(x) = f(x).$$

For the moment let's take

$$\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} + a\frac{du}{dx} + b\,u$$

where a and b are constants, i.e. the so-called constant coefficient problem.

The general solution to homogeneous problem

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + b\,u = 0$$

can easily be found by looking for a solution having the special form $u(x) = e^{rx}$,

$$0 = \frac{d^2u}{dx^2} + a\frac{du}{dx} + b\,u = (r^2 + a\,r + b)\,e^{rx} \quad \Rightarrow \quad r^2 + a\,r + b = 0.$$

The polynomial on the right is called the *characteristic polynomial* for the constant coefficient problem. Its roots can be found by using the quadratic formula

$$r^2 + a\,r + b = 0 \quad \Rightarrow \quad r = \tfrac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right).$$

There are three cases you must be ready for. They are:

(1) The roots are real and distinct; $a^2 - 4b > 0$.

(2) The roots are distinct but complex; $a^2 - 4b < 0$.

(3) The characteristic polynomial has only one root; $a^2 - 4b = 0$.

I'll illustrate these three cases by considering the following three examples.

The first case:

(1) $$\frac{d^2u}{dx^2} + 2\,\frac{du}{dx} - u = 0 \quad \Rightarrow \quad r^2 + 2r - 1 = 0 \quad \Rightarrow \quad r = -1 \pm \sqrt{2}.$$

This yields two solutions: $u(x) = e^{-x}\, e^{\sqrt{2}x}$ and $u(x) = e^{-x}\, e^{-\sqrt{2}x}$. Therefore, the general solution to example (1) is

$$u(x) = e^{-x}\left(c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}\right).$$

As I stressed to you in class, I suggest you note that by redefining constants as follows

$$\frac{\tilde{c}_1 + \tilde{c}_2}{2} \equiv c_1, \quad \frac{\tilde{c}_1 - \tilde{c}_2}{2} \equiv c_2,$$

the solution above can be written in the alternate form

$$u(x) = e^{-x}\left(\tilde{c}_1 \cosh(\sqrt{2}x) + \tilde{c}_2 \sinh(\sqrt{2}x)\right).$$

The second case:

$$(2) \qquad \frac{d^2u}{dx^2} + 2\,\frac{du}{dx} + 2\,u = 0 \quad \Rightarrow \quad r^2 + 2r + 2 = 0 \quad \Rightarrow \quad r = -1 \pm \sqrt{-1} = -1 \pm i.$$

This yields two solutions: $u(x) = e^{-x}\, e^{ix}$ and $u(x) = e^{-x}\, e^{-ix}$. Therefore, the general solution is

$$u(x) = e^{-x}\left(c_1 e^{ix} + c_2 e^{-ix}\right).$$

But by again redefining constants

$$\frac{\tilde{c}_1 - i\tilde{c}_2}{2} \equiv c_1, \quad \frac{\tilde{c}_1 + i\tilde{c}_2}{2} \equiv c_2,$$

together with Euler's formula, we can rewrite the general solution to example (2) as

$$u(x) = e^{-x}\left(\tilde{c}_1 \cos(x) + \tilde{c}_2 \sin(x)\right).$$

The third case:

$$(3) \qquad \frac{d^2u}{dx^2} + 2\,\frac{du}{dx} + u = 0 \quad \Rightarrow \quad r^2 + 2r + 1 = 0 \quad \Rightarrow \quad r = -1 \pm \sqrt{0} = -1.$$

This yields only one solution: $u(x) = e^{-x}$. However, as you saw on your previous homework, in this double root case a second solution is found by multiplying the first solution by x. That is $u(x) = xe^{-x}$ is a second solution. Therefore, the general solution to example (3) is

$$u(x) = e^{-x}\left(c_1 + c_2 x\right).$$

Please memorize how to solve the constant coefficient, homogeneous, second order problem by finding the root(s) of its characteristic polynomial and then identifying which case above the roots fall into, i.e. case (1) two real and distinct roots, case (2) two distinct complex roots and case (3) a single double root.

We'll now focus on techniques for finding u_p, the particular solution for the inhomogeneous problem.

For the constant coefficient ODE, *the method of undetermined coefficients*, or what I call *the method of guessing*, is a fast and probably the easiest way to find a particular solution. Here we'll solve

$$\frac{d^2u}{dx^2} + a\,\frac{du}{dx} + b\,u = f(x) \quad (a \text{ and } b \text{ are constants})$$

for certain right hand sides:

$$(1)\;\; f(x) = x^n \;(n = 0, 1, 2, \ldots), \quad (2)\;\; f(x) = e^{hx}, \quad (3)\;\; f(x) = \sin(hx) \text{ or } \cos(hx).$$

(See `https://wikipedia.org/wiki/Method_of_undetermined_coefficients` for others.) The key to employing this technique is to guess the correct form for u_p.

(1) Given $f(x) = x^n$, try

$$u_p(x) = x^s\left(\alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0\right)$$

(typically $s = 0$ but you might need $s = 1$ or 2).

Use $s = 0$ unless $u = 1$ solves the homogeneous ODE. Use $s = 1$ unless $u = x$ solves the homogeneous ODE. Use $s = 2$ if both $u = 1$ and $u = x$ solve the homogeneous ODE.

Here are two examples.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + u_p = x^2. \quad \text{Try } u_p = \alpha_2 x^2 + \alpha_1 x + \alpha_0\,.$$

Here we can take $s = 0$ since $u = 1$ does not solve the homogeneous problem. Plug in the trial and equate powers of x

$$x^2 = \frac{d^2u_p}{dx^2} + u_p = (2\alpha_2) + (\alpha_2 x^2 + \alpha_1 x + \alpha_0) = \alpha_2 x^2 + \alpha_1 x + \alpha_0 + 2\alpha_2$$
$$\Rightarrow \quad \alpha_2 = 1,\; \alpha_1 = 0,\; 2\alpha_2 + \alpha_0 = 0 \quad \Rightarrow \quad \alpha_2 = 1,\; \alpha_1 = 0,\; \alpha_0 = -2$$

So, $u_p = x^2 - 2$.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + \frac{du_p}{dx} = x^2. \quad \text{Try } u_p = x\,(\alpha_2 x^2 + \alpha_1 x + \alpha_0)\,.$$

Here we take $s = 1$ since $u = 1$ does solve the homogeneous problem but $u = x$ does not. Plug in and equate powers of x

$$x^2 = \frac{d^2u_p}{dx^2} + \frac{du_p}{dx} = (6\alpha_2 x + 2\alpha_1) + (3\alpha_2 x^2 + 2\alpha_1 x + \alpha_0)$$
$$\Rightarrow \quad 3\alpha_2 = 1,\; 6\alpha_2 + 2\alpha_1 = 0,\; 2\alpha_1 + \alpha_0 = 0 \quad \Rightarrow \quad \alpha_2 = 1/3,\; \alpha_1 = -1,\; \alpha_0 = 2.$$

So, $u_p = x^3/3 - x^2 + 2x$.

(2) Given $f(x) = e^{hx}$ $(h \neq 0)$, try

$$u_p(x) = x^s \alpha\, e^{hx} \quad (\text{typically } s = 0 \text{ but you might need } s = 1 \text{ or } s = 2).$$

Use $s = 0$ unless $u = e^{hx}$ is a homogeneous solution. Use $s = 1$ unless $u = xe^{hx}$ is a homogeneous solution. Use $s = 2$ otherwise.

Here are two examples.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + u_p = e^x. \quad \text{Try } u_p = \alpha e^x\,.$$

Here we take $s = 0$ since $u = e^x$ does not solve the homogeneous problem. Plug in and equate

$$e^x = \frac{d^2u_p}{dx^2} + u_p = \alpha e^x + \alpha e^x = 2\alpha e^x \quad \Rightarrow \quad \alpha = 1/2.$$

So, $u_p = \frac{1}{2}e^x$.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} - u_p = e^x. \quad \text{Try } u_p = x\,\alpha e^x\,.$$

Here we take $s = 1$ since $u = e^x$ does solve the homogeneous problem. Plug in and equate

$$e^x = \frac{d^2u_p}{dx^2} - u_p = \alpha(xe^x + 2e^x) - \alpha xe^x = 2\alpha e^x \quad \Rightarrow \quad \alpha = 1/2.$$

So, $u_p = \frac{1}{2}x\,e^x$.

(3) Given $f(x) = \sin(hx)$ or $\cos(hx)$ $(h \neq 0)$, try

$$u_p(x) = x^s\,(\alpha\cos(hx) + \beta\sin(hx)) \quad \text{(typically } s = 0 \text{ but you might need } s = 1\text{)}.$$

Use $s = 0$ unless $u = \sin(hx)$ and $\cos(hx)$ solve the homogeneous ODE. If they do take $s = 1$.

Here are two examples.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + \frac{du_p}{dx} = \sin(x). \quad \text{Try } u_p = \alpha\cos(x) + \beta\sin(x)\,.$$

Here we take $s = 0$ since $u = \sin(x)$ and $\cos(x)$ do not solve the homogeneous problem. Plug in and equate

$$\begin{aligned}\sin(x) = \frac{d^2u_p}{dx^2} + \frac{du_p}{dx} &= (-\alpha\cos(x) - \beta\sin(x)) + (-\alpha\sin(x) + \beta\cos(x))\\ &= (\beta - \alpha)\cos(x) + (-\beta - \alpha)\sin(x). \quad \Rightarrow \quad \alpha = -1/2,\ \beta = -1/2.\end{aligned}$$

So, $u_p = -\frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)$.

Find a particular solution to the following.

$$\frac{d^2u_p}{dx^2} + u_p = \sin(x). \quad \text{Try } u_p = x\,(\alpha\cos(x) + \beta\sin(x))\,.$$

Here we take $s = 1$ since $u = \sin(x)$ and $\cos(x)$ do solve the homogeneous problem. Plug in the trial and equate

$$\begin{aligned}
\sin(x) &= \frac{d^2u_p}{dx^2} + u_p \\
&= (-\alpha x\cos(x) - \beta x\sin(x) - 2\alpha\sin(x) + 2\beta\cos(x)) + (\alpha x\cos(x) + \beta x\sin(x)) \\
&= -2\alpha\sin(x) + 2\beta\cos(x) \quad \Rightarrow \quad \alpha = -1/2,\ \beta = 0.
\end{aligned}$$

So, $u_p = -\frac{1}{2}x\cos(x)$.

The method of guessing works great when it applies, i.e. it only works for constant coefficient problems with certain right hand sides $f(x)$. Be very careful however to use the correct form for your trial solution. I suggest you first solve the homogeneous problem and use its solution to determine how to modify the form of your trial particular solution.

In exercises 1 – 5, use the *method of guessing* to find the **general solution** to the given inhomogeneous, linear and constant coefficient differential equations.
Hint: To solve part (d) use linearity together with your answers from parts (a) – (c).

1. $\dfrac{d^2u}{dx^2} + 2\dfrac{du}{dx} - 3u = f(x)$ where

   (a) $f(x) = x^2 + 1$ (b) $f(x) = \sin(x)$

   (c) $f(x) = e^x$ (d) $f(x) = x^2 + 1 + 5\,e^x$

2. $\dfrac{d^2u}{dx^2} + 4u = f(x)$ where

   (a) $f(x) = 2x + 1$ (b) $f(x) = \sin(x)$

   (c) $f(x) = \sin(2x)$ (d) $f(x) = 2\sin(x) + 3\sin(2x)$

3. $\dfrac{d^2u}{dx^2} - 9u = f(x)$ where

   (a) $f(x) = x^2$ (b) $f(x) = \sin(3x)$

   (c) $f(x) = e^{-3x}$ (d) $f(x) = 9x^2 + 10\,e^{-3x}$

4. $\dfrac{d^2u}{dx^2} - 2\dfrac{du}{dx} + u = f(x)$ where

   (a) $f(x) = 3x^2 + 4$ (b) $f(x) = e^x$

   (c) $f(x) = e^{2x}$ (d) $f(x) = 6x^2 + 8 + 5\,e^x + 6\,e^{2x}$

5. $\dfrac{d^2u}{dx^2} = f(x)$ where

(a) $f(x) = 1$ (b) $f(x) = x$

(c) $f(x) = x^2$ (d) $f(x) = 6x^2 + 2x + 4$

Next I'll state a method for finding a particular solution to the inhomogeneous, second order, linear ODE based on a general idea known as Duhamel's principle; Jean-Marie Duhamel: `https://wikipedia.org/wiki/Duhamel's_principle`.

Consider the second order and linear differential operator

$$\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)\,u,$$

where this time the coefficients $a(x)$ and $b(x)$ do not need to be constant. Solve the homogeneous initial value problem

$$\mathcal{L}(v) = 0 \ \text{ with initial conditions } \ v(z) = 0, \quad v'(z) = 1.$$

Note here the initial conditions are specified at $x = z$. I'll denote the solution by $v(x; z)$. Now, the integral

$$u_p(x) = \int_a^x v(x; z) f(z)\, dz$$

is the solution to the inhomogeneous ODE

$$\mathcal{L}(u_p) = f(x) \ \text{ where } u_p \text{ satisfies } \ u_p(a) = u_p'(a) = 0.$$

Note how the integral's lower limit a appears in the initial conditions for u_p above.

Here's a constant coefficient example. It's easy to compute (you do it) that the solution to the homogeneous IVP

$$\frac{d^2v}{dx^2} + v = 0, \quad v(z) = 0, \ v'(z) = 1, \quad \text{is} \ \ v(x; z) = \sin(x - z).$$

Therefore, according to Duhamel's integral formula

$$u_p(x) = \int_0^x \sin(x - z)\, \sin(z)\, dz \ \text{ solves } \ \frac{d^2u_p}{dx^2} + u_p = \sin(x) \ \text{ with } \ u_p(0) = u_p'(0) = 0.$$

(I took the lower limit $a = 0$ for convenience only.) To evaluate the integral, use

$$\sin(a)\sin(b) = \tfrac{1}{2}\left(\cos(a - b) - \cos(a + b)\right),$$

and compute

$$u_p(x) = \int_0^x \sin(x - z)\sin(z)\, dz = \tfrac{1}{2}\int_0^x (\cos(x - 2z) - \cos(x))\, dz = \tfrac{1}{2}\sin(x) - \tfrac{1}{2}x\cos(x).$$

Note that $u_p(0) = u_p'(0) = 0$. Compare this to the particular solution found in the method of guessing example at the top of page five.

Here's a variable coefficient example.

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}-\frac{1}{x^2}v=0,\quad v(z)=0,\ v'(z)=1,\quad\Rightarrow\quad v(x;z)=\frac{1}{2}\left(x-\frac{z^2}{x}\right).$$

This variable coefficient differential equation comes from a class of variable coefficient problems known as *Cauchy–Euler* equations. I'll show you how to solve these later. Therefore,

$$u_p(x)=\int_1^x \frac{1}{2}\left(x-\frac{z^2}{x}\right)\sqrt{z}\,dz \quad\text{solves}\quad \frac{d^2u_p}{dx^2}+\frac{1}{x}\frac{du_p}{dx}-\frac{1}{x^2}u_p=\sqrt{x}.$$

(I took $a=1$ to avoid division by zero.) Assume $x>0$ and evaluate the integral to get

$$u_p(x)=\frac{4}{21}x^{5/2}-\left(\frac{x}{3}-\frac{1}{7x}\right).$$

Note that $u_p(1)=u_p'(1)=0$.

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6. Use Duhamel's integral formula to find a particular solution to the following inhomogeneous problems. (Take your lower limit $a=0$.)

(a) $\dfrac{d^2u}{dx^2}+u=\cos(x)$ (c) $\dfrac{d^2u}{dx^2}=x^2$

(b) $\dfrac{d^2u}{dx^2}-u=e^{-x}$ (d) $\dfrac{d^2u}{dx^2}-\dfrac{du}{dx}=e^x$

Answers: (a) $u(x)=\frac{1}{2}x\sin(x)$. (b) $u(x)=\frac{1}{2}\sinh(x)-\frac{1}{2}xe^{-x}$. (c) $u(x)=\frac{1}{12}x^4$. (d) $u(x)=xe^x-e^x+1$.

7. Use Duhamel to find a particular solution and then determine the general solution for

$$\frac{d^2u}{dx^2}-2\frac{du}{dx}+2u=\sin(x).$$

Your integral should look like $u(x)=\int_a^x e^{x-z}\sin(x-z)\sin(z)\,dz=\ldots$ The integration here may take some effort.

8. Do the same as exercise 6. You may assume $x>0$ and take your lower limit $a=1$.

(a) $\dfrac{d^2u}{dx^2}+\dfrac{1}{x}\dfrac{du}{dx}=x$ (b) $\dfrac{d^2u}{dx^2}-\dfrac{1}{x}\dfrac{du}{dx}=x$

You may freely use the facts that the general solution to the homogeneous problems are $c_1+c_2\log(x)$ for part (a) and $c_1+c_2x^2$ for part (b).
Answers: (a) $u(x)=-\frac{1}{3}\log(x)+\frac{1}{9}(x^3-1)$. (b) $u(x)=\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{6}$.

Here's a terse verification of the Duhamel formula for the variable coefficient problem.

The Leibniz integral rule is a slight extension of the fundamental theorem of calculus. It states

$$\frac{d}{dx}\int_a^x f(x,y)\,dy=f(x,x)+\int_a^x\frac{\partial}{\partial x}f(x,y)\,dy.$$

The Leibniz rule is pretty easy to derive. See your calc3 text or ask me to derive it in class.

Apply the Leibniz rule to the Duhamel formula

$$\frac{d}{dx}u_p(x) = \frac{d}{dx}\int_a^x v(x;z)\,f(z)\,dz = v(x;x)\,f(x) + \int_a^x \frac{\partial}{\partial x}v(x;z)\,f(z)\,dz.$$

Now, the initial condition $v(z) = 0$ is shorthand for $v(x;z)|_{x=z} = 0$. Since this is true for any z, we must have $v(x;x) = 0$. Therefore

$$\frac{d}{dx}u_p(x) = \int_a^x \frac{\partial}{\partial x}v(x;z)\,f(z)\,dz.$$

Apply the Leibniz rule to this again

$$\frac{d^2}{dx^2}u_p(x) = \frac{\partial}{\partial x}v(x;x)\,f(x) + \int_a^x \frac{\partial^2}{\partial x^2}v(x;z)\,f(z)\,dz.$$

The initial condition $v'(z) = 1$ is shorthand for $(d/dx)\,v(x;z)|_{x=z} = 1$. Since this is true for any z, conclude this time that $(\partial/\partial x)\,v(x;x) = 1$. Therefore

$$\frac{d^2}{dx^2}u_p(x) = f(x) + \int_a^x \frac{\partial^2}{\partial x^2}v(x;z)\,f(z)\,dz.$$

Plug these into the differential equation to get

$$\begin{aligned}&\frac{d^2u_p}{dx^2} + a(x)\,\frac{du_p}{dx} + b(x)\,u_p \\ &= f(x) + \int_a^x \left(\frac{\partial^2}{\partial x^2}v(x;z) + a(x)\,\frac{\partial}{\partial x}v(x;z) + b(x)\,v(x;z)\right) f(z)\,dz.\end{aligned}$$

Finally, since v solve the homogeneous equation, the bracketed term in the integral above is identically zero. Therefore

$$\frac{d^2u_p}{dx^2} + a(x)\,\frac{du_p}{dx} + b(x)\,u_p = f(x) + 0 = f(x).$$

How to choose the lower limit a

From the previous paragraph we have shown

$$u_p(x) = \int_a^x v(x;z)\,f(z)\,dz, \quad u_p'(x) = \int_a^x \frac{\partial}{\partial x}v(x;z)\,f(z)\,dz.$$

As you all know, $\int_a^a \cdots = 0$. Therefore, our Duhamel solution will automatically satisfy

$$u_p(a) = 0, \quad u_p'(a) = 0.$$

Here's an example to show how the choice of lower limit can be exploited to solve the IVP. Solve the inhomogeneous IVP

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = \sqrt{x}, \quad \text{with initial conddитions } \ u(1) = 2,\ u'(1) = 3.$$

In a previous example, we found

$$u_p(x) = \int_1^x \frac{1}{2}\left(x - \frac{z^2}{x}\right)\sqrt{z}\,dz = \frac{4}{21}x^{5/2} - \left(\frac{x}{3} - \frac{1}{7x}\right).$$

Check that as stated above, we must have $u_p(1) = u_p'(1) = 0$. Later we'll see that the general solution to this homogeneous Cauchy–Euler equation is

$$u_h(x) = c_1 x + c_2 x^{-1}.$$

Therefore the general solution to the inhomogeneous problem is

$$u(x) = c_1 x + c_2 x^{-1} + \left(\frac{4}{21}x^{5/2} - \left(\frac{x}{3} - \frac{1}{7x}\right)\right).$$

Now let's use the initial conditions to determine c_1 and c_2.

$$\begin{aligned} 2 &= u(1) = c_1 + c_2 + 0 \\ 3 &= u'(1) = c_1 - c_2 + 0 \end{aligned} \quad \Rightarrow \quad c_1 = 5/2,\ c_2 = -1/2.$$

So, the given inhomogeneous IVP's solution is

$$u(x) = \frac{5}{2}x - \frac{1}{2}x^{-1} + \frac{4}{21}x^{5/2} - \left(\frac{x}{3} - \frac{1}{7x}\right).$$

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9. Use your Duhamel results from exercise 6 to solve the following inhomogeneous IVPs.

   (a) $\dfrac{d^2u}{dx^2} + u = \cos(x),\ u(0) = 1,\ u'(0) = 2.$

   (b) $\dfrac{d^2u}{dx^2} - u = e^{-x}, \quad u(0) = 1,\ u'(0) = 2.$

10. Use your Duhamel results from exercise 8 to solve the following inhomogeneous IVPs.

   (a) $\dfrac{d^2u}{dx^2} + \dfrac{1}{x}\dfrac{du}{dx} = x,\ u(1) = 1,\ u'(1) = 2.$

   (b) $\dfrac{d^2u}{dx^2} - \dfrac{1}{x}\dfrac{du}{dx} = x,\ u(1) = 1,\ u'(1) = 2.$

Recall the homogeneous solutions to part (a) and (b) are given in exercise 8.
