Math 3331 Exam 2. Sanders Fall 2023

This exam has five problems, and all five will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, last name first, and student id number on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless indicated otherwise.

1. Factorize the following second order differential operators into the composition of two first order operators. You are not asked to solve a differential equation here.
   (a) \( \frac{d^2 u}{dx^2} + \frac{du}{dx} - 6u \).
   (b) \( \frac{d^2 u}{dx^2} + 4u \).

2. Determine the general solution (homogeneous solution + particular solution) to each of the following by using the method of guessing.
   (a) \( \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} = e^x \).
   (b) \( \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} = x \).
   (c) \( \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} = \sin(x) \).
   (d) \( \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} = x + e^x \).

Since the LHs are all the same, you only need to find the homogeneous solution once. Also, you may reuse your results from parts (a)–(c) to answer part (d).

3. Use Duhamel to find the solution of each of the following initial value problems.
   (a) \( \frac{d^2 u}{dx^2} + u = x, \ u(0) = u_x(0) = 0 \).
   (b) \( \frac{d^2 u}{dx^2} = e^x, \ u(0) = u_x(0) = 0 \).

4. Write each of the following scalar differential equations as a first order system.
   (a) \( \frac{d^2 u}{dt^2} + 2 \frac{du}{dt} - u = 0 \).
   (b) \( \frac{d^2 u}{dt^2} - u \left( \frac{du}{dt} \right)^2 = 0 \).
   (c) \( \frac{d^2 u}{dt^2} = 1 \).
   (d) \( \frac{d^3 u}{dt^3} = \frac{d^2 u}{dt^2} + u \).

5. Consider the matrix \( A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} \).
   (a) Determine the eigenvalues of \( A \).
   (b) Determine the eigenvectors of \( A \).
   (c) Determine the matrix \( e^{At} \).
   (d) Use part (c) to solve the following initial value problem:
      \[
      \begin{align*}
      \frac{du}{dt} &= -3u + 4v, \quad u(0) = 1, \\
      \frac{dv}{dt} &= -2u + 3v, \quad v(0) = 0.
      \end{align*}
      \]