1. Solve the following Cauchy-Euler IVP’s.

(a) \( x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} - u = 0 \), \( u(1) = 0 \), \( u'(1) = 1 \).

(b) \( x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + u = 0 \), \( u(1) = 1 \), \( u'(1) = 0 \).

(c) \( \frac{d^2 u}{dx^2} + \frac{3}{x} \frac{du}{dx} + \frac{1}{x^2} u = 0 \), \( u(1) = 1 \), \( u'(1) = 1 \).

(d) \( \frac{d^2 u}{dx^2} - \frac{1}{x} \frac{du}{dx} + \frac{1}{x^2} u = 0 \), \( u(1) = 0 \), \( u'(1) = 1 \).

2. Use Duhamel’s method to solve the following.

(a) \( \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u = x \), \( u(1) = 0 \), \( u'(1) = 0 \).

(b) \( \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u = x^2 \), \( u(1) = 0 \), \( u'(1) = 0 \).

Answers: (a) \( u(x) = \frac{x^3 - 2x + x^{-1}}{8} \). (b) \( u(x) = \frac{(2x^4 - 5x + 3x^{-1})}{30} \).

Duhamel’s method can also be used to solve the following first order inhomogeneous system

\[
\frac{dx}{dt} - \lambda x = f(t), \quad x(0) = 0 \quad \Rightarrow \quad x(t) = \int_0^t e^{A(t-\tau)} f(\tau) \, d\tau,
\]

where \( x, f \in \mathbb{R}^d \) and \( A \in \mathbb{R}^{d \times d} \) is a constant matrix.

3. Solve the following inhomogeneous system.

\[
\begin{align*}
\frac{dx}{dt} - y &= t, \quad x(0) = 0, \\
\frac{dy}{dt} - x &= e^t, \quad y(0) = 0.
\end{align*}
\]

Hint: Recall

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.
\]

If a 2 \times 2 matrix \( A \) only has one eigenvalue, then it is either already diagonal, or it can’t be diagonalized. There is however, in this latter case, an invertible matrix \( S \) such that

\[
S^{-1} AS = J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.
\]

\( J_\lambda \) is called the Jordan form for the non-diagonalizable matrix \( A \). As shown in class, once \( S \) is determined we can easily evaluate

\[
e^{At} = Se^{J_\lambda t}S^{-1}, \quad \text{where} \quad e^{J_\lambda t} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]
Here’s a recipe to determine $S$. First, compute an eigenvector, say $r_\lambda$, associated to $A$’s
eigenvalue $\lambda$. Next, determine any vector, say $g$, which is not parallel to $r_\lambda$. That is, $g$
can be any nonzero vector which is not an eigenvector. Compute the vector $r = (A - \lambda I)g$.
Since $(A - \lambda I)^2 = 0$, note that $r$ will be an eigenvector of $A$. Finally, let $S$ be the $2 \times 2$
matrix whose first column is $r$ and second column is $g$. This is a similarity transformation $S$
which will do the trick.
Here’s an example. Let $A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$. This matrix has only one eigenvalue, $\lambda = 3$, and
a one dimensional eigenspace spanned by $r_\lambda = (2, 1)^t$. Let’s pick $g = (1, 0)^t$ since it’s simple and not an eigenvector. Compute
$$
 r = (A - \lambda I)g = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.
$$
(Note that $r$ above is an eigenvector.) Form the matrix $S$ with columns $r$ and $g$
$$
 S = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow S^{-1}AS = J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.
$$
Again let me stress, this construction works only for $2 \times 2$ non-diagonalizable matrices.

4. Let $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$. This matrix is not diagonalizable. Determine the Jordan form $J_\lambda$
for $A$ and a similarity transformation $S$ so that $S^{-1}AS = J_\lambda$.
Partial answer: $J_\lambda = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$.

5. For the matrix $A$ in the previous exercise, determine $e^{At}$.
My answer: $e^{At} = e^{4t} \begin{pmatrix} 1 - 2t & 4t \\ -t & 1 + 2t \end{pmatrix}$.

6. Solve the initial value problem
$$
\frac{dx}{dt} = 2x + 4y, \quad x(0) = 1,
$$
$$
\frac{dy}{dt} = -x + 6y, \quad y(0) = 2.
$$